



**Australian Government**  
**Department of Defence**  
Defence Science and  
Technology Organisation

# On Modelling Hybrid Uncertainty in Information

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DSTO-RR-0325

## **ABSTRACT**

Numerical induction models are considered in this report to be models that aggregate lower-level information into higher-level measures for decision making. Various forms of uncertainty may be present in such models including hybrid uncertainties within the information elements being aggregated. After a review of some existing approaches for representing higher-order uncertainty in information, a new approach is presented to enable greater fidelity of uncertainty representation, and consequently more rigorous uncertainty management in aggregation operations. Several different applications then demonstrate the proposed procedures which have direct relevance to many Defence decision making models where higher-order uncertainty is ubiquitous. The overall objective of these procedures is to extract as much meaning from the input information as possible.

## **RELEASE LIMITATION**

*Approved for public release*

*Published by*

*Command and Control Division  
DSTO Defence Science and Technology Organisation  
PO Box 1500  
Edinburgh South Australia 5111 Australia*

*Telephone: (08) 8259 5555  
Fax: (08) 8259 6567*

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AR-013-848  
February 2007*

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## Executive Summary

Many uncertainty modelling formalisms are limited when it comes to representing hybrid uncertainty combinations and this weakness is especially relevant to numerical induction models where lower-level information is aggregated, or synthesised, into higher-level global values. A fundamental problem is that most of these formalisms do not discriminate between the different uncertainty sources that can be present, and the different forms of uncertainty that they can introduce. Consequently, these methods tend to be generalised approaches at the theoretical level with practical limitations for representing real-world information characteristics. Another problem is that many of these generalised theoretical approaches are based on restrictive assumptions which are not always explicit. This report presents an alternative approach which commences with the initial identification of the different sources of uncertainty that may be present, and then maps from the combination of these various sources to a composite uncertainty representation. The methodology proposed is founded on a new interpretation of some concepts that already exist in conventional Fuzzy Set and Possibility Theory.

The primary modification made to conventional Fuzzy Set Theory is to distinguish between the ambiguous and vague forms of fuzzy uncertainty. New measures are defined for fuzzy ambiguity and vagueness, which are then compared with some existing fuzzy uncertainty measures. Several applications of the proposed uncertainty modelling methods are presented such as determining measures of typicality for information with various hybrid uncertainty combinations, and a fuzzy set representation which also includes some statistical information for sparse data sets. An example of complex system reliability analysis with hybrid information forms is also used to demonstrate how the results derived using the proposed methods can deviate significantly from estimates derived by classical Fuzzy Set Theory which does not distinguish between the different uncertainty forms arising from different sources. Such a significant discrepancy in a measure upon which high-level decisions are made could then greatly affect a decision.

The objective of the proposed methodology is to imbue more theoretical rigour into measure aggregation computations through improved fidelity in modelling the uncertainty forms that are actually present in the component data. By doing so the outputs from such numerical induction models used for decision making should then be more meaningful, and the resulting decisions be more valid. A variety of Defence decision making models are potentially affected by this new approach to uncertainty representation including models for planning capability development, for project risk evaluation, for integration of diverse forms of intelligence, and cost/benefit/risk models in general.

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## Glossary

Ambiguity:	A measure of Ambiguity
$ A $ :	Cardinality of set A
$\tilde{A}$ :	A fuzzy variable A
COA:	Centre of Area
COG:	Centre of Gravity
CUI:	Composite Uncertainty Interval
$\Delta(A)$ :	E-Possibility of A (of Dubois and Prade) as the lowest possibility that an interval and all sub-intervals belong to A
EXOR:	Exclusive OR operator
FN:	A Fuzzy Number
FEV:	The Fuzzy Expected Value
FST:	Fuzzy Set Theory
FTA:	Fault Tree analysis
GKF:	Globular Knowledge Fusion
HUMINT:	Human Intelligence
MTBF:	Mean Time Between Failure
$\mu_A(x)$	Membership value of x belonging to fuzzy set A
MTBR:	Mean Time Between Repair
NSP':	Non-Specificity measure
PI:	Performance Index
PMxU:	Principle of Maximum Uncertainty
Poss or $\Pi$ :	A possibility value
P or Prob or Pr:	A probability value
PN:	A possibilistic Number
Rel:	Reliability
SFQ:	The stochastic fuzzy quantity
SFS:	Structure Function Synthesis
T:	Total Uncertainty %
TPD:	A triangular possibility distribution
TpPD	A trapezoidal possibility distribution
U:	Uncertainty
Vagueness:	A measure of vagueness

# 1. Introduction

Within many intelligent decision support systems there are computational processes for information synthesis of information with several kinds of uncertainty. Data sets may contain varying degrees of information completeness; from rich statistical sufficiency in data streams of crisp values, through less complete statistical information of fuzzy values, to sparse information on fuzzy values. The information forms addressed in this report are data sets where the fuzzy uncertainty forms dominate the statistical forms of uncertainty. In this report “soft” information will be defined to be information associated with some form of fuzziness, including any probabilistic information not derived from statistical sufficient samples of evidence. The fuzzy uncertainty forms that are the focus of this report may exist in the information elements themselves, or be introduced by conflicts between data elements, or by data acquisition methods, or by methods to summarise or condense imperfect data sets, or by changeable contexts. Various types of decision analysis models used in Defence may also process such diverse types of soft information including models for planning capability development, for project risk evaluation, for integration of non-homogeneous forms of intelligence, and cost/benefit/risk models in general.

Most authors in the field of Fuzzy Set Theory do consider fuzzy sets to be sets based on “vague” concepts, and especially linguistic concepts. However, an examination of the literature demonstrates that there are no precise definitions to distinguish “vagueness” from another term “ambiguity” which is also widely used to describe fuzzy uncertainty. Generally speaking, there are a variety of understandings for both these terms, but overall in both the probabilistic and fuzzy literature, ambiguity is taken to refer to a multiplicity of values induced by insufficient or missing information. On the other hand, vagueness as a higher-order form of uncertainty is mainly discussed in the fuzzy literature as “graded” belief, with the Measure of Fuzziness most commonly used for its quantification. Thus, vagueness is most commonly evaluated by the “fuzziness” of a fuzzy set. However, this type of understanding, and level of discrimination severely limits the ability of classical fuzzy set theory to manage composite uncertainty forms that can be induced from a wide variety of sources. Although Klir [56] has proposed a framework, called “Generalised Information Theory”, which to a degree lays some foundations for identifying combinations of different types of uncertainty in information, the categories in that framework only address vagueness to a fairly limited degree. Joslyn and Rocha [49] have also attempted to establish a framework to identify diverse uncertainty combinations. However, beyond a taxonomy, they also present no practical methods for representing and measuring the uncertainty in the different combinations of uncertainty types.

Although differentiating between ambiguity and vagueness does not in general affect fuzzy deductive inferencing very significantly, as in rule-based control systems, it has a much greater impact on numerical induction models where low level information is combined into higher level global values. Such models are widely used for complex decision analysis, system reliability analysis, complex operations evaluation, and cost/benefit/risk analysis in general. This report attempts to improve the fidelity of modelling hybrid uncertainty forms which may be present in imperfect information so that more meaning can be associated with the outputs of numerical induction models.



To commence, a framework is first defined for identifying the different types of uncertainty that may be embedded in a model and its information inputs. Using the possibility concept for belief measurement, it is then demonstrated how all soft uncertainty variants can be reduced to combinations of ambiguity and vagueness. The primary proposal of this report is that a new dimension of possibility is required to adequately model the hybrid forms of uncertainty that can exist within a body of soft evidence or information with hybrid uncertainties. This proposal is supported by a detailed analysis of existing fuzzy uncertainty measures, and a comparison of their behaviour with that of two new measures for the separate and distinct forms of fuzzy uncertainty: ambiguity and vagueness. Simply speaking, vagueness is defined as the presence of vague set elements in a fuzzy set, something that may appear obvious, but is not reflected in current understandings of vagueness. In the terminology of fuzzy set theory, this has taken the focus away from the so-called “Type” of a fuzzy set for representing higher-order uncertainty, and redirected it towards the “Level” of a fuzzy set, which is used here to represent the uncertainty about the set elements themselves.

In light of the new notions introduced, extensions to several existing fuzzy uncertainty measures are also developed to enable those classical measures to be applied to uncertainty measurement when vagueness is present. For example, an extension to an existing formulation of fuzzy entropy is introduced using a derivation from first principles. Also, within the general field of numerical measure aggregation, the summarisation of sets of imperfect soft information is a common requirement. However, a very wide range of competing techniques appear in the literature for condensing soft information and this can be somewhat confusing for practitioners. To address this potential confusion, several methods that have been proposed for determining measures of typicality for soft information are categorised on the basis of the forms of hybrid uncertainty that they actually address or best suit. Finally, the management of hybrid uncertainty forms using the proposed representations is demonstrated for complex system reliability synthesis. The results of the proposed methods for this example are also shown to deviate significantly from those of several approaches that are based on more traditional fuzzy set theory techniques.

While it may not seem to be very important to distinguish between ambiguity and vagueness in many computational processes, this report attempts to show that the distinction can significantly affect a numerical induction model’s output if they are separately identified and carefully managed in the information aggregation processes. Overall, the need for clear identification of the various sources of uncertainty in the information inputs to models is emphasised so that the intrinsic nature of the fuzzy variables induced can be clearly determined. Section 2 examines some existing approaches to fuzzy uncertainty representation and measurement. Section 3 then introduces a new approach to fuzzy uncertainty representation, and Section 4 demonstrates some new fuzzy uncertainty measures. Section 5 then extends some existing fuzzy uncertainty measures for vagueness when it is present, and Section 6 applies the proposed notions for determining measures of typicality for sets of soft information. Finally, Section 7 demonstrates the new approach for determining the reliability of a complex system with soft information inputs.

## 2. Existing Approaches to Fuzzy Uncertainty Analysis

### 2.1 Introduction

Many authors have observed that there is widespread disagreement in the literature regarding the meaning of different uncertainty (U) terminology. Toth [94] has noted that 25 different terms can be readily identified, Smets [90] has offered a thesaurus on uncertainty, and Pal and Bezdek [76] have presented a summary of fuzzy U measures noting some divergent use of terminology. The term “uncertainty” in general refers to incompleteness or inexactness of information of some sort. But there can be many different types of inexactness in soft information leading to a variety of questions such as:

- What is ambiguity and vagueness and how do they differ?
- Does vagueness = fuzziness ?
- Is non-specificity always caused by some kind of variability?
- What does fuzziness mean?
- How can conflict be measured?
- Are there different types of imperfect belief?

However, U due to random variations is obviously different from U due to the vague definition of a label. And the difficulty of identifying and representing the different U forms that are present increases significantly as the different forms compound, especially when higher-order forms of U are present (uncertainty of uncertainty).

Several U taxonomies or typologies can also be found, such as those by Smithson [91], Klir and Folger [54], and Pal and Bezdek [76]. Among these it can be noted that “vagueness” is sometimes identified as a U form distinct from “fuzziness”, but invariably without any precise description of the difference. Non-specificity is generally attributed to imprecision or inexactness, as the degree to which there is more than one point value or singleton. It is usually identified as one main form of fuzzy U, while “fuzziness” is considered to be the other main form of fuzzy U. But there are considerable differences of opinion regarding how to represent and subsequently quantify, combinations of fuzzy U and randomness. Nevertheless, there is fairly general agreement that the “Specificity” concept relates to the multiplicity of possible values. And while definitions of measures to evaluate the “Fuzziness” of a fuzzy set are fairly consistent, as the similarity of a set to its complement, there seems to be no clear understanding on what this actually means from the point of managing U in computations. To address this weakness, precise and functional definitions of ambiguity and vagueness will be presented in this report, and used to identify the *information* associated with a fuzzy variable. Based on a literature survey, an overview of the meaning that other authors attribute to these two terms will be presented initially to highlight the lack of practicable definitions for U management in numerical induction.

It is not uncommon in the literature to equate ambiguity with the U caused by statistical variations (Mares [67] for example). Others equate ambiguity with fuzziness. (Pal [77] “fuzziness is the average ambiguity in a fuzzy set”). Vagueness is most commonly equated with fuzziness as the lack of distinction between a set and its complement (Klir and Folger [54], Dubois and Prade [27]). Mares [67] also views ambiguity to be U caused by statistical

variations which probability theory can address, and vagueness to be the general U form represented in fuzzy sets of which there are three aspects:

- the uncertainty of “mere existence” as the maximum degree of membership
- the degree of dispersion of the set
- the degree of irregularity of the set.

However, none of the above interpretations for vagueness can *distinguish* subjective vagueness from simple inexactness. Regarding this form of subjective uncertainty, Pal [77, p. 99] in a recent summary of fuzzy U measures has also noted that “...some research is required on the subjective assessment of U associated in a system”, i.e. of subjective assessments of systems’ state variables. In general, the two widely accepted fuzzy U measures: Yager’s Specificity [108,109] and Higashi and Klir’s Non-Specificity [46] (as quantified by the U-Uncertainty measure), will be shown in this report to be quantifications of ambiguous U i.e. related to the presence of multiple possible set values. Although there are a variety of other measures to quantify this form of U, most relate to the multiplicity of something, whatever way that something or set element is defined. Regarding vagueness, some notions in the literature which could be interpreted as vagueness, although not necessarily called vagueness, will now be summarised. As indicated previously, most authors do not consider the concept of vagueness to be a distinct form of U different to fuzziness and requiring a special definition. Moreover, it is important to examine the usage and interpretation given to vagueness by Dubois and Prade who are the major contributors to the field of possibility theory along with the founder Lotfi Zadeh [117]. Among their many works (such as [23-30]) that refer to vagueness, [28, 29] will be used as a basis for the following comments.

The second reference [29] provides insight to their general conceptualisation of vagueness. In it they describe fuzzy sets as “convenient fictions”, which implies a qualitative interpretation whereby the distribution’s boundary should not be considered as being exact or unique. They associate vagueness with “graduality” whereby  $x$  values on the real axis (say  $x$  = Age in years) cannot be precise in the canonical fuzzy equation  $x = \tilde{A}$  (say  $\tilde{A}$  = Old). They also point-out that a fuzzy set may have several different interpretations depending on the sources of U (likelihoods, randomness, subjectivity, lack of information) and that the presence of vagueness can determine the suitable interpretation. Unfortunately, they do not describe exactly what such a special interpretation may be. In the other reference [28], they introduce a new kind of possibility called “E-Possibility” for “Everywhere-possibility”. This is defined by means of a fuzzy interval set as follows. Whereas traditional possibility  $\Pi(A)$  represents the maximum possibility that an element belongs to A, E-Possibility  $\Delta(A)$  represents the lowest possibility that an *interval*, and all *sub-intervals* (i.e. all point-values within the interval), belong to A. They state that E-Possibility is a more pessimistic, or conservative, estimate based on the stronger membership requirement. Consequently, it represents a kind of guaranteed value of possibility which can be used as a lower limit. These two values then effectively define a band of possibility for the distribution. They also state [28,p50] that the stronger requirement implies  $\Pi(A) \geq \Delta(A)$  for all A. Although the justification for this statement is not obvious to this author, it is deemed to be a logical requirement to allow a continuous band to exist. Dubois and Prade then explore the implications of these upper and lower possibilities on rule-based inferencing and define four semantic variations of gradual rules with “vague relationships” between conditions and conclusions. At first sight, E-Possibility, as the lowest

possibility of a set of values, seems to be quite close to the vagueness concept to be presented in this report. However, there are some key differences. One is that for Dubois and Prade the two different possibility values *always* exist and always define a band of U at the boundary. In contrast, vagueness as presented in this report does not always exist and is solely determined by the U sources. Furthermore, their strong requirement  $\Pi(A) \geq \Delta(A)$ , does not hold in fact for all A. Figure 5b (to be presented in section 3) clearly illustrates that the possibility of each  $\alpha$ -cut interval  $\Delta(A)$ , is not always less than  $\Pi(A)$  i.e. the band of U at the fuzzy distribution boundary may have cross-over points where it is zero. In summary, the dominant usage of vagueness by Dubois and Prade would appear to be for “graduality” as represented by imprecise divisions or quantisations of a fuzzy variable. Although they use the E-Possibility concept to describe a “vague” relationship between conditions and conclusions of a fuzzy rule, this does not describe vagueness within a fuzzy set itself.

Next, the meaning attributed to vagueness by Kruse et al. will be elicited from [64] which surveys various methods for systematic U management in knowledge-based systems. That monograph investigates vagueness as it applies to variables which are measurable and point-valued, such as “distance”. Their viewpoint is that vagueness is induced around a variable’s value when information is combined that arises from different contexts, where each contextual source can introduce a set of possible values due to variations caused by subjective or approximate measurements i.e. either of these. As they state, this approach is an extension of Moore’s [70] interval algebra since it utilises a layered set of intervals (L-sets): “we conceive an L-set to be induced by unknown imprecise descriptions in an unknown number of contexts”. So the meaning they attribute to vagueness is a general form of measurement imprecision caused by contextual variations.

K.Y.Cai [11] has also introduced a notion which seems to relate to vagueness. This is the Q-Scale measure which he states “is slightly different from a crisp valued possibility measure”. However, at the present incomplete stage of development of his conceptual framework, a more exact definition is not available. For the Q-Scale measure to be equivalent to the definition of this report, the non-crisp possibility measures Cai describes would need to be associated with indivisible intervals, as opposed to intervals representing a range of values. But Cai [10] also alerts us to a very early author in the development of fuzzy set theory who presented a notion very similar in spirit to what the Q-Scale measure hints at. That reference was Nahmias [71] who in 1978 suggested that fuzzy sets could be interpreted from the uncertainty metric axis rather than from the real axis, calling this a “fuzzy variable”. In effect, from this viewpoint a fuzzy set is interpreted as a set of consonant intervals, in the manner of a vague variable to be defined in this report. However, it seems that Nahmias could not provide a good reason why this viewpoint should be adopted unconditionally. Perhaps for this reason it was not developed into any practical or functional techniques; although it did precipitate some related theoretical analysis.

Two factors could be relevant to that situation. The first was that applications using rule-based fuzzy logic were the primary focus at that time, and Nahmias' idea seemed to have little relevance to that application. The second point was that Nahmias did not present it as a conditional concept that was only invoked by certain types of situations, but rather presented it in the same manner as the always present E-possibility of Dubois and Prade [28] was conceived. Consequently, it did not appear to have any special benefits and so did not find any significant applications in fuzzy set theory. However, if Nahmias' fuzzy variables are linked to certain higher-order U sources they would be very similar to the vague variables proposed later in this report.

Another interpretation of vagueness that appears throughout the literature is to consider it to be that form of U represented by Type 2 or qualitative fuzzy sets, with an indistinct boundary defined by upper and lower membership limits. Turksen's [96] interval-valued fuzzy sets are examples of this type of vagueness. A variety of other interval-valued fuzzy sets exist, and Sambuc [79] in 1975 provides what may be the earliest example of an application using interval sets. Certain methods used to derive the indistinct boundary limits are also very similar in principle to the belief and plausibility limits in the Theory of Evidence. For example, Atanassov [3] in 1986 introduced "Intuitionistic" fuzzy sets with boundaries determined from evidence for the degree of *membership* in a set, and the degree of *non-membership* in a set. Type 2 interval-valued fuzzy sets and Intuitionistic fuzzy sets could also be interpreted as one type of vague fuzzy set. The final example of a notion similar to fuzzy vagueness has been introduced by Wong and Wang [106]. In that work the authors attempt to develop a qualitative framework for a non-numerical assessment of uncertainty. However, the method is developed by analogy with axiomatic treatments of numerical *ambiguity* measures (such as Fishburn's [35] to be discussed in the next section). Consequently, it is more a qualitative assessment of the ambiguity type of uncertainty.

The above examples have been presented to illustrate the range of understandings for ambiguity and vagueness that can be found in the literature. Perhaps Termini's [93] investigation into the epistemological problems encountered when defining vagueness is a suitable place to end this section. He discusses what various philosophical authors have written about the subject and points-out that vagueness is induced by *both* information limitations residing in the object, plus external information factors such as different viewpoints or contexts. He also states that what is required is a *formal framework* to accommodate *both* these different sources of vagueness. The framework, or taxonomy, that will be presented in this report has been designed to embrace both of Termini's vagueness sources plus some additional sources. In general, U measures are useful for ensuring valid propagation of U through operations in knowledge-based decision systems, especially where the soft input information includes hybrid forms of U and subjective human inputs. U measures can also be useful for completing possibility distributions when the soft evidence yields ambiguous possibility information. In this case, the possibility distribution may be specified by applying the Principle of Minimum Uncertainty, which maximises the information on Uncertainty using the notion that the information content is the dual of the uncertainty content (more information means less U). Next, some axiomatic treatments of fuzzy U measures will be described.

## 2.2 General Axiomatic Investigations

Many authors have adopted the approach of identifying candidate uncertainty measures by the degree to which they satisfy several (usually about six) general axiomatic requirements. Ebanks [31] and Pal [77] demonstrate this approach. Such axioms generally describe limits of behaviour plus some essential or distinguishing characteristics of a particular type of U measure. It is apparent that this approach is also subject to semantic problems and diverse understandings. Some examples of this type of literature will now be summarised.

### Ambiguity

Fishburn [35] in "On the Theory of Ambiguity" has provided a set of axioms in relation to a special type of ambiguity. Yager [112] has subsequently showed that, in fact, most existing measures of fuzzy U satisfy Fishburn's axiomatic requirements. In that work, Fishburn clearly demonstrates the semantic problems existing in the field and expresses some vacillation on the descriptor most appropriate for his target form of U. In the end, Fishburn adopts the U divisions of Savage [80] based on the U originating from:

- Relativity of judgment
- Ambiguity/Vagueness in the judgment of a value
- Judgment of relative likelihoods
- Ambiguity/Vagueness around the likelihood judgments

Restated more concretely, these sources of U might be: scale bias, measurement imprecision, statistical variability, and evidence incompleteness. Fishburn identifies his subject as the final class and expresses ambivalence on whether to use ambiguity, or vagueness, for that kind of U (as fuzzy likelihoods). Fishburn states that he finally chooses "ambiguity" because Ellsberg's [32] notion of ambiguous probabilities has been widely accepted in decision theory. This is rather interesting because his initial intention was "to stand aside from explicit ties of event ambiguity to probability". So just as Savage does not distinguish ambiguity from vagueness, neither does Fishburn due to a lack of the necessary conceptual notions. As an example of the opposite choice, Fishburn mentions Wallsten [101] who argues that the subject of fuzzy likelihoods would better be described by the term "vagueness" (rather than "ambiguity"). Both ambiguity and vagueness for Fishburn refer then to uncertain likelihoods, as multiple possible probabilities due to the insufficiency of evidence.

### Vagueness

Overall, a survey of the literature revealed no axiomatic treatments of vagueness measures, as a distinct form of fuzzy uncertainty, although some measures that could relate to vagueness will be subsequently described.

### The Fuzzy Characteristic

Ebanks[31] provides a thorough set of axioms for Measures of Fuzziness and the degree to which a measure satisfies these axioms is sometimes used to indicate the strength or weakness of *any* U measure. However, it does not seem prudent to extrapolate beyond the intended domain because some of Ebanks' requirements cannot be assumed to be *universally* true. For example, the maximality requirement states that an U *measure* is maximum at:

$$\mu_A(x) = 0.5 \forall x \in X, \text{ where } \mu_A(x) \text{ is the degree of belief that } x = \tilde{A}.$$

While this value represents the degree of maximum belief uncertainty, it is not true that an aggregate *U information measure* (traditional or new species) must be a maximum. This point relates to a fundamental question concerning the effect of the maximum belief level ( $\mu^*$ ) in non-normal fuzzy sets on any type of aggregate *U* measure. This effect is subsequently demonstrated in Section 4, which compares several *U* measures, where it is shown that even the traditional Measure of Fuzziness does not satisfy this maximality constraint unless it is modified. But more importantly, the maximality requirement cannot be assumed to apply to *information* measures of fuzzy uncertainty.

### General Information Measures

Forte and Kampe de Fariet [34] in 1967 developed axiomatic definitions for general information measures of crisp events in a probabilistic event space. Bertoluzza et al [5] subsequently extended this for a fuzzy event space by aggregating  $\alpha$ -cut information over the fuzzy variable representation. Generally speaking, the minimum value of an information measure is zero, and the maximum value is the size of the universe of discourse. For quantifications of *U information*, the application of the Principle of Minimum Uncertainty with information measures then maximises the information available when there is a choice. However, Bertoluzza's [5] treatment proceeds at a very general level and does not distinguish between different types of fuzzy *U*.

## 2.3 Measures of Fuzzy Uncertainty Forms

To facilitate this survey of quantifications or measures of fuzzy uncertainty, the difference between *two distinct types* of measure that relate to fuzzy *U* in a fuzzy set should first be clarified. Understanding this difference is fundamental to the correct propagation of *U* in systemic computations.

### Uncertainty Information Measures

This class of measure is the main subject of this report and in fuzzy set theory set membership degree, or value possibility, are the scales used to assess the uncertainty of set elements. On the basis of these units, this class of measure quantifies the *lack of* information that results from the different types of fuzzy uncertainty. They are thus measures of the forms of fuzzy information and are useful for ensuring coherent information propagation through computations.

### Measures of the Fuzzy Characteristic

The so-called Measures of Fuzziness relate to the graded shape of the characteristic function of partial belief that in aggregate is a fuzzy set or possibility distribution. A variety of definitions for this "fuzziness" exist but most are based on distance measures between a fuzzy set and its complement. These measures assess the fuzzy dispersion characteristic and provide information on the "belongingness" property of elements to a fuzzy set. Because very different fuzzy sets can have the same value for the Measure of Fuzziness these measures do not quantify information about fuzzy *U*. They only describe a fuzziness feature, which is the amount of *grade* in a fuzzy set (to be demonstrated in Section 4.3).

### 2.3.1 Ambiguity Measures

A wide variety of U measures based on the cardinality of a fuzzy set have been proposed. In the terminology of this report, this type of U measure is restricted to measuring ambiguity since it is based on the choice of a single value from a set of multiple possible values. A concise summary of such measures, many of which are based on extensions to the probabilistic concept of entropy, has been provided by Pal and Bezdek [76]. Two of the most common measures are:

Specificity :                      Yager [109] definition -  $S(\Pi) = \int_0^{\alpha^*} \frac{1}{|A|} d\alpha$                       (1)

U- Uncertainty:                      Higashi and Klir [46] definition based on Hartley Information concept [44] -

$$U\text{-Uncertainty} = \int_0^1 \log_2 |A| d\alpha \quad (2)$$

where  $|A|$  is the Cardinality at  $\alpha$  and  $\alpha^*$  is the max  $\mu$  or  $\pi$

These two measures are thus derived from functions of set cardinality over different levels of belief. For continuous possibility distributions, the cardinality is calculated as  $(1+\alpha\text{-cut interval})$  and thus the minimum cardinality value (where cardinality includes values at both ends of an interval) of all  $\alpha$ -cuts is at the modal value. Regarding U-Uncertainty, Klir [57, p19] has recently recommended that the measure for non-normal sets be normalised by dividing the maximum degree of belief ( $\mu^*$ ) to form a "meaningful weighted average". The effect of this procedure will be demonstrated in the following U measure comparisons (Section 4.3) where it is shown to be of questionable validity since it violates the Principle of Maximality. Both Specificity and U-Uncertainty have their own strengths and weaknesses (also as discussed in Section 4.3) and neither measure can be considered to be superior. For example, Yager's Specificity, as an inverse function of set cardinality is conceptually clear, but one could ask why it should be a *non-linear* hyperbolic function (inverse) as opposed to a linearly decreasing function. Also, Klir and Higashi's U-Uncertainty definition effectively reduces a non-binary fuzzy event space to a binary event space to apply the Hartley information notion, which again seems to be somewhat questionable.

The final measure to be introduced, which could possibly be interpreted as fuzzy ambiguity, comes from the theory of complex systems. It is termed the "radius of information" [107]. But on close scrutiny, this is not a measure of fuzzy uncertainty forms because it addresses a variability characteristic of a set of information. For this reason, it is similar to a measure of statistical ambiguity, rather than one for evaluating the ambiguity form of fuzzy uncertainty.

### 2.3.2 Vagueness Measures

Some U measures will now be described that could be interpreted as vagueness measures. In the first, Wierman [105] defines an explicit vagueness measure based on the  $\alpha$ -cut set of elements with possibility measures  $\leq$  the  $\alpha$  level.

$$V(\tilde{A}) = \int_0^1 \log_2 |A_\alpha^\leq| d\alpha \quad (3)$$



As Wierman himself states, it is a variation on the Hartley information measure. Thus, being based on the identification of a single value from a possible set of alternative values, it represents another type of ambiguity measure and cannot distinguish any higher-order forms of fuzziness that may be present. Another attempt at quantifying vagueness has been made by Chen [17], although these are not actually U measures. Chen has proposed measures of *similarity* for “vague” sets having boundaries defined by evidence for, and against, a proposition. These measures would also appear to be related to the Intuitionistic sets of Atanassov [3]. Burillo and Bustince [8] have also developed entropy-like measures for both Intuitionistic and interval-valued sets. Both of these measures thus relate to the fuzzy set boundary type of vagueness, which is quite distinct to the higher-order form of possibility existing in vague possibility distributions as is proposed in this report.

The final measure [18] to be introduced concerning vagueness quantification is called the “measure of information diffusion”. However, this measure also relates to a spread of information as does the “radius of information” [107]. Furthermore, for this measure the many assumptions and input parameters required limit its practical value to a large extent.

### 2.3.3 Measures of Fuzziness

Measures of Fuzziness are primarily functions of the degree of non-committal belief that exists in the fuzzy set and represent the degree to which the membership values differ from 0 or 1 [23,54]. But it could be asked just what kind of uncertainty information does this degree of non-committal belief represent. These measures are based on a key feature of fuzzy sets which is that the intersection of a fuzzy set and its complement yields another fuzzy set. Among many definitions for a Measure of Fuzziness, a widely accepted version is the Yager [108] definition, which aggregates the difference between these two finite sets using the Hamming distance, as the absolute value of the difference. For discrete finite sets the definition is:

$$f(A) = \sum_{x \in X} [1 - |\mu_A(x) - c(\mu_A(x))|] \quad (4)$$

$$\text{where } c(\mu_A(x)) = 1 - \mu_A(x)$$

Higashi and Klir [46] have generalised the Yager measure for a range of definitions for the complement. Significantly, the standard definition for complement does not clearly distinguish the fuzziness between non-normal and normal fuzzy sets. But for the purpose of subsequent measure comparisons, we will use the modified definition below for the set complement. (The results shown in Table 6 (as Mod.) will highlight the difference this modification makes.)

$$c(\mu_A(x)) = \mu_{\max} - \mu_A(x) \quad \text{instead of } c(\mu_A(x)) = 1 - \mu_A(x)$$

As previously noted, Measures of Fuzziness represent the number of support values that are effectively unsure, or uncommitted, and actually measure a characteristic of the distribution rather than the fuzzy information content.

Klir [57] also notes: “*whether it is meaningful to view a reduction in fuzziness as a gain in information depends on the accompanied change in nonspecificity*”.

For this reason, Measures of Fuzziness by themselves are not appropriate for the management of U propagation in computations. Another separate problem is that many definitions are somewhat insensitive. The comparative examples that follow illustrate this as well as demonstrating how different definitions measure slightly different characteristics of non-committal belief. For example, using the traditional definition of complement, the same magnitude for the Hamming metric can be derived from a variety of different sets. In effect, it normalises all sets and is solely a function of the support size (the fuzzy set modal value also having no effect). However, the modified version exhibits a different behaviour. In addition to support size, it is a function of the total grade (slope) in a fuzzy set which is also a function of the maximum membership value. Here also, the modal value has no effect since it only affects the left and right portions but not the total amount of grade present. Moreover, for measures of lack of belief, the problem context itself should determine which version of the Measure of Fuzziness is most relevant. For example, the maximum level of belief may have little credibility in reality due to the high levels of U in the set's derivation. In such cases, the modified gradient sensitive version would be difficult to justify.

### 2.3.4 Approaches to Higher-Order Uncertainty Representation

The term “hybrid uncertainty” refers to mixtures of fundamentally different forms of U. For example, U induced by stochastic variations, sparse data, approximate measurement, subjective ratings, and so on. The primary focus of this report is modelling of hybrid U i.e. how to represent mixtures of different U forms. The general modelling approach proposed is to clearly identify each separate U source, the doors where U can enter into the information pertaining to a variable, and the composite U representation is then determined by the highest form of component U. For convenience, here “higher-order U” will be assumed to mean “hybrid U”. However, the term “higher-order” U is often used to refer to compounds of the U metrics used, rather than to the actual forms of U present. For example, second-order U may be probability of probability, possibility of possibility, or possibility of probability. Before discussing higher-order U further, several different approaches to modelling and representing hybrid U will be surveyed from a practical perspective.

Oblow [72] develops a hybrid U representation method for combining fuzzy sets with the upper and lower probability measures of Evidence Theory. In this case, probabilistic U is combined with fuzzy ambiguity. Kaufmann and Gupta [53] have also proposed the concept of “hybrid numbers” consisting of a fuzzy part and a probabilistic part. But as will be discussed in more detail later, their representation only allows an incomplete range of algebraic operations. In a similar fashion, Ferson and Ginzburg [33] have proposed an approach for hybrid arithmetic, for combinations of fuzzy and probabilistic information, but in their case for the full range of algebraic operations. Finally, Joslyn and Rocha [49] have described a general taxonomy of hybrid uncertainty based on arranging many combinations of elements, sets, and intervals with partial belief measures. However, the manner by which the different uncertainty combinations may actually arise, and their composite meanings, are not described. These two factors are essential for adequate U management in computations. In general, none of the above authors provide a mapping method from U sources to the net type of U induced in the hybrid U information. Consequently, this limitation tends to restrict the usefulness of the above methods for the systems modeller; in addition to their inability to distinguish ambiguity from vagueness.

Concerning the modelling of higher-order U in the fuzzy domain, three existing approaches can be distinguished. The first is the representation of fuzzy probabilities of which there are many examples, especially in the literature of fuzzy reliability analysis. The modelling of fuzzy probabilities itself presents no insurmountable obstacles to the existing constructs of fuzzy set and possibility theory. The second approach is to model fuzzy measures of fuzzy variables, as in fuzzy random variables. But in the opinion of this author, such modelling attempts are severely constrained by the previously described limitations in modelling the effects of different U combinations. This, in turn, is caused by the inability to represent and distinguish between different types of fuzzy U. Considerable confusion can be attributed to this limitation, as is apparent in the field of possibilistic reliability analysis [10,12] that will be discussed in more detail in Section 6. The third approach to modelling higher-order U in the fuzzy domain is to use Type 2 fuzzy sets with inexact fuzzy boundaries. Recently, Yang and Liu [113] have proposed a method to represent a double fuzzy variable induced by the fuzzy perception of a fuzzy variable, using a Gaussian membership function with fuzzy parameters i.e. a fuzzy bounded possibility distribution. But this type of inexact fuzzy distribution does not really constitute a true second-order U representation, although Type 2 fuzzy representations of probability values would be a true second-order U representation.

In the probabilistic community, higher-order U representations have been used also to model ambiguity. Although this report concerns the modelling of ambiguity and vagueness in the fuzzy domain, some probabilistic usage will now be briefly described. Frisch and Baron [38] have provided a concise survey of the different understandings of “ambiguity” among the probabilistic community, covering the influential viewpoints of Savage [80], Ellsberg [32], and Gardenfors and Sahlin [40]. They conclude that beyond describing the general U introduced by stochastic variations, ambiguity is represented as inexact or multiple probabilities, and is a result of missing information. They also suggest that subjective probabilities can be differentiated from objective probabilities in this light, thus resolving a longstanding representational problem within the probabilistic community. Thus, according to Frisch and Baron [38], probability *intervals* capture ambiguity caused by subjective evaluations. (Subsequently, an alternative approach will be introduced to distinguish between subjective evaluations and the ambiguous probabilities that can be induced by sparse data.) Overall, for higher-order U modelling in the probability framework, three types of extensions to classical probability theory can also be identified. The first type of extension to capture higher-order U could be considered to be the use of probability intervals indicating upper and lower limits [99,100]. The second approach to modelling higher-order U could be considered to be the Dempster-Shafer Theory of Evidence [83], which is also derived from the notion of imprecise probabilities, but in the form of two new measures with more specific meanings called the belief and plausibility measures. The third approach is to model second-order probability as probability of a probability, and this approach is conceptually similar to the second dimension of possibility to be introduced in this work. Although there are considerable information demands for estimating second-order probabilities, Chavas [16] provides an example of that less common probabilistic approach. It is also interesting to note that many authors using probabilistic methods implicitly assume that one formalism alone must be applied to model all forms of U. To this author this seems to be an unnecessary and inexplicable constraint.

## 2.4 Section Summary

A variety of fuzzy U concepts and measures that exist in the literature have been described. The most widely accepted fuzzy U measures are Specificity, U-Uncertainty, and Measures of Fuzziness of several variants. Due to their set cardinality foundations, Specificity and U-Uncertainty have been shown to be ambiguity measures quantifying the degree of elemental multiplicity inherent in a fuzzy set. However, ambiguity was seldom explicitly used in the literature to describe “multiplicity”. Although Measures of Fuzziness were generally considered to address a fundamental feature of fuzzy sets which was the similarity of a fuzzy set to its complement, what this actually means from the point of U management was not clear. Pal and Bezdek [76] have also noted that Measures of Fuzziness are often combined with ambiguity type measures to form an evaluation of the total fuzzy U, which is of questionable value due to the dissimilar and incompatible units. Although a few authors did present notions that seemed to relate to “vagueness”, some with accompanying measures, most of these related to the fuzzy boundary of Type 2 fuzzy set, which is of limited practical value for U management in numerical induction. And while considerable effort in the literature has been focused on methods to quantify Type 2 uncertainty around the set boundary or belief distribution, less attention has been paid to the quantification of the Level 2 form of uncertainty. Nevertheless, some notions were discovered that appeared quite similar in spirit to the distinct form of fuzzy U to be described as vagueness in this report (even though they were not necessarily referred to as “vagueness”). Those authors (Nahmias, Dubois and Prade, and Cai) also intimated that a separate non-standard type of possibility could also be interpreted from a fuzzy set. Overall, the various axiomatic treatments of fuzzy uncertainty appear to be hindered by semantic problems, and are not sufficiently specific in their characterisations to lead to results that could be applied over a wide variety of formulated uncertainty measures. In summary, combinations of different forms of U arising from different sources of U can result in hybrid composites that require some form of net U representation. It is suggested in this report that the fidelity by which a U modelling formalism can represent the composite U forms present, can have an impact on the robustness of a model’s computations. Nevertheless, a comprehensive approach for mapping from hybrid U combinations to net U representations could not be discovered in the literature survey. And most importantly, no definitions could be found that would enable the difference between ambiguity and vagueness to be modelled, such that different fuzzy U forms could be carefully measured and managed in numerical induction computations.

### 3. A New Approach to Fuzzy Uncertainty Representation

#### 3.1 Introduction

The approach to U modelling that will be adopted focuses first on the identification of the component sources of U in each variable. A classification taxonomy of the potential sources of U is then used to identify the composite level of U present in each variable. While the number of U sources is quantified by the proposed U Dimension concept, the composite form of U is determined by the forms of possibility introduced by each U source. Using a new dimension of potential possibility to capture the vagueness form of fuzziness, each variable in a model is then identified as one of three types: crisp number, fuzzy number (FN), or possibilistic number (PN). The PN represents a set of consonant intervals of a type different to other fuzzy interval representations. To place these fuzzy U forms of information into context, the levels of U present in the general modelling process will first be decomposed.

#### 3.2 Levels of Uncertainty in Modelling

The use of a conceptual model for computational analysis imputes several levels of potential uncertainty. This section will briefly outline these levels to provide the context for the main subject which is the analysis of inherited uncertainty in information sets. In the following schema, higher numerical levels represent lower levels of conceptual uncertainty.

**Level 1:**            Uncertainty in Objective or Problem Definition --

The highest level of U is U of purpose, or understanding what the problem actually is. This may be related to limitations of personal perceptions, confusion, lack of information, or problem complexity. Personal factors such as experience, skill, and bias can influence the cognisance of a problem. Certain analytical methods such as personal construct theory, cognitive mapping, or the soft systems methodology may assist in consolidating a problem's definition. Varying interpretations of an abstract concept may also contribute to this level of uncertainty. If the key characteristics of a problem are not identified correctly the chances of finding a useful or valid solution are unlikely.

**Level 2:**            Uncertainty in Model Conceptualisation --

At this level a *conceptual model* is adopted as the computational framework and U exists due to the fit of a model to the problem. A range of questions must be answered to develop the model and fit it to reality in an adequate manner (fitness to purpose). Such considerations are:

What paradigm ?        Hierarchical or complex system model, single or multiple models.

Example paradigms are neural networks, fuzzy cognitive maps, influence diagrams, or Bayes Nets.

What structure?        Where are the boundaries, how detailed, and what is the appropriate granularity of the model?

**Level 3: Uncertainty in Computational Macro-Structure --**

U induced by the adequacy of *model components* such as the inferencing or clustering technique, type of nodal squashing function, type of aggregation operator, or belief propagation mechanism.

**Level 4: Uncertainty in Computational Micro-Structure --**

U due to the adequacy of the *component details* of a technique, such as parameters in neural-net squashing function gain or bias, information aggregation optimism/pessimism degree, or some parametric sensitivity determinant.

**Level 5: Uncertainty in Sample Evidence --**

This U level pertains to the estimation of a single measure for a variable, as with a measure of typicality, from a *multiplicity of information* (i.e. a set) for that variable. This is to be distinguished from the induction of a measure from data relating to a collection of other variables, which would represent a structural form of U. There are three different types of Level 5 U. The first is U due to *stochastic variations*. The second is associated with the *validity* of information elements themselves. This does not mean measurement error (to be included in Level 6 U), but rather, the conflict or contradictions originating at the source of the information. The third type of Level 5 U is related to the *amount of evidence* available upon which the estimate is based. This may vary from a very sparse set to a continuous stream of data. As will be subsequently detailed, the presence of these different types of U in a set of data contributes to the composite U form in the estimate.

**Level 6: Intrinsic Uncertainty--**

These are the pre-computational forms of U inherited from both the definition of a variable and any measurement error or estimation limitations. Rather than being a function of the multiplicity of information available, this U level refers to constraints on information content by virtue of the concept of the variable itself and its measurement.

The above U levels provide the highest classification categories of U sources that will be presented in this report. For convenience, these six levels can be further separated into two groups: Levels 1-4 as modelling structure U, and Levels 5-6 as information U. Since the focus of this report is on U forms in information elements, rather than U induced by the model and its mechanisms, Level 5 and 6 will now be decomposed further into those hybrid sources that may be present. A mapping method will then be proposed from these hybrid sources of U to the composite U form they induce. This mapping will be based on the different types of possibility that each hybrid may induce.

### 3.3 Sources of Fuzzy Uncertainty in Variables

The separate sources of U within the evidential and intrinsic U Levels 5 and 6 will now be itemised for the purpose of identifying the forms of fuzziness that can be induced through those hybrid U combinations. The various combinations of these separate sources of U will subsequently be discussed in Section 3.5.

### 3.3.1 Intrinsic Uncertainty

Intrinsic U forms (Level 6) may be introduced into an information element by the conceptual definition itself, or by measurement or estimation.

#### *Variable Definition Uncertainty:*

Uncertainty may be introduced by vague or ambiguous definitions. For convenience, two types of definition will be distinguished: numerically quantitative definitions and qualitative definitions. Some examples of U associated with variable definition follow.

Quantitative Definition -	Numerical ambiguity
* Approximate Size-	X = Opposing force strength, Weapon effect magnitude
Qualitative Definition -	Epistemic vagueness or impossibility of knowing exactly
* Semantic vagueness:	Linguistic variables
	X = Group morale, Joint Force Preparedness, Plan riskiness
*Hypothetical vagueness:	It is impossible to know now exactly what a future value will be.
	X = inflation rate next year, missile stocks in 5 yrs time
	X = number of eggs Hans can eat, number of students in a mini-minor (in a normal or special situation?)

An unknown or flexible context results in a flexible universe of discourse and possibilistic vagueness where multiple indistinguishable values may exist at any  $\alpha$ -cut of a fuzzy set. An unknown universe may be invoked by modelling assumptions, as exemplified when modelling the safety or reliability of a nuclear power plant. If engineering experience and industry statistics are used to define the universe of discourse, a bounded universe is invoked where probabilities (as crisp or fuzzy numbers) can represent event likelihoods. However, if the modeller allows the possibility of inconceivable or freak type of events not seen before, an unknown universe is implied where event likelihoods are vague possibilities.

#### *Measurement Uncertainty:*

Measurement U can be divided into qualitative or quantitative measurement as in the following examples.

Quantitative measurement-	(Objective numerical quantification)
* Approximate - inaccuracy of scale, inconsistent measuring device,	or rough estimation
Qualitative Measurement-	
* Subjective evaluation -	(Human estimation with possible bias)
	It feels like 22°C, I think the probability of a link failing is 0.15.
	From experience 95% survive about one year, I say a rating of 3 on a scale of 10.
* Qualitative relation between a measured value and the target variable --	
	Another form of qualitative measurement can occur when the relationship between the actual variable and the measured quantity is indirect, or

associational as a covariate. Evaluating the likelihood of Joseph being overweight from his age is such a case.

Linguistic variables with cognitive uncertainty usually incorporate a high degree of semantic vagueness, and for this reason it is unreasonable to impute crisp evaluations even from a collection of opinions.

### 3.3.2 Sample Evidence Uncertainty

In addition to the above forms of intrinsic U in a set of information pertaining to a variable, there may also be the three types of sample Level 5 U previously described. However, since this report is only concerned with the representation of fuzzy U forms, modelling stochastic U is outside its scope.

For the purpose of identifying the composite forms of fuzziness induced, a set of information will also be classified according to the amount of statistical information present in a set of data:

Statistically sufficient sample - enough information for parametric statistical estimation

Statistically insufficient sample - incomplete information resulting in some degree of fuzziness

The net effects of various hybrid combinations of U will be examined in detail in Section 3.5.

Table 2 summarises the potential hybrids of intrinsic and evidential fuzzy U forms, whereby two types of fuzzy numbers (to be described) may be induced when there is insufficient information in the set to legitimately develop crisp parametric statistical estimates. Other approaches to classifying U sources are possible but for the purpose of relating U sources to composite U forms, the decomposition given above is considered to be sufficient. Section 6 will discuss several techniques for determining a measure of typicality for different types of soft information based on the hybrid U forms present.

## 3.4 Basic Classes of Variables

It is postulated that the type of fuzzy quantity induced, and whether fuzzy ambiguous U or fuzzy vagueness U is dominant, is determined by the hybrid combinations of U sources present. For the purpose of describing how two types of fuzzy quantity can be induced, the meaning of the possibility measure will first be discussed.

### 3.4.1 The Meaning of the Possibility Measure

Zadeh [117] first introduced the notion of possibility as a higher-order U interpretation of a fuzzy set. Dubois and Prade [25] developed the concept further and pointed-out [27] that there may be two interpretations in common language of the expression “possible values”. The first meaning can be considered as that of “capacity” as in “it is possible *for* Hans to eat 10



eggs.” This they call the physical interpretation of what *can* occur. The second meaning relates to available evidence as in “it is possible *that* it will rain tomorrow”. This they call the logical interpretation estimating what *may* occur. Smets [82] also identifies these two types of possibilities calling them the “physical” and “epistemic” possibilities respectively. Smets notes the similarity with two types of probabilities: “aleatory” reflecting chance and “epistemic” reflecting credible belief. The correspondence with the so-called objective and subjective probabilities is obvious. But in this text possibility will be taken to mean the plausibility of an event or value. In this sense, the concept relates to the very *existence* of the item and the measure describes the degree to which it *can exist*; usually termed the “plausibility”, meaning that it is not impossible. This interpretation equates to the physical meaning of the above authors of what can occur. This is also the meaning that Dubois and Prade [24] suggest is the appropriate one for possibility theory. However, the other “may be” meaning will be discussed in Section 3.5.4 on higher-order U representations where it is suggested to be a hybrid of probability and possibility estimates.

In contrast to the possibility measure, the probability measure itself describes the belief degree to which something *will exist*, or *will eventuate* based on some *a priori* knowledge, real as evidence, or imagined as belief. It should also be noted that the concept of a “typical value” can be applied to both types of belief estimation. The ontological distinctions between probability and possibility can be summarised as follows:

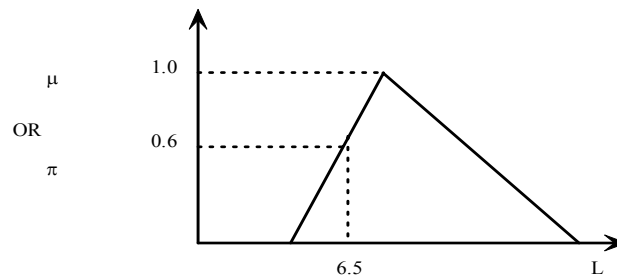
- (i) *can* exist, as possibility or plausibility  
 $\pi = 1$ , means the event or value can definitely exist  
 $\pi = 0$ , means the event or value definitely cannot exist (no plausibility).
- (ii) *will* exist, as probability evaluated from past behaviour  
 $pr = 1$ , means the event or value certainly will occur  
 $pr = 0$ , means the event or value definitely will not occur (no chance).

Furthermore, probability itself as the likelihood that something will occur may also be estimated using the possibility measure (that the likelihood value *can* occur). Such fuzzy possibility distributions of the probability measure may then be of two types (to be defined shortly) depending on the nature of the evidence. It is also obvious that it must be possible for an event to exist before it will occur and this is the simple basis of the Consistency Principle of Zadeh [117] which states that a possibility measure must be  $\geq$  than the probability measure for the same event. From this Principle, a possibility measure could be considered as an upper probability estimate. However, De Cooman [21] has shown recently that a special kind of lottery is required for the possibility measure to be equivalent to an upper probability. De Cooman [21] has also made a pertinent comment about the lack of a clear interpretation for the possibility measure, which this section attempts to address:

*“To understand the practical meaning of the possibility measure i.e. how should they be used in practical reasoning, we need a behavioural interpretation that relates the possibility measure to decisions and actions.”*

The possibility measure is also subtly different to another fuzzy set measure: the degree of membership ( $\mu$ ) of an element in a set. The following example is presented to illustrate how the interpretations for possibility distributions and fuzzy sets differ, i.e. how the meaning embedded in each is different. (This difference is in addition to a basic difference in the

complementation operation caused by the higher-order of uncertainty measured by the possibility metric and which will be discussed later.) Although the difference described below may seem somewhat trivial, it is an important distinction because operations on the possibility measure relate more to uncertainty management than to relational operations on the membership grade or fuzzy set boundary. This means that for some possibility distributions many mathematical operations, including the sup-min implementation of the Extension Principle, may be of questionable value. In short, the set membership values ( $\mu$ ) can frequently be interpreted as truth values and used for managing inferencing operations on the fuzzy set boundaries that indicate the degree to which values belong to a set. In contrast, under the fuzzy calculi of relations as used in information synthesis,  $\pi$  transforms should be managed primarily by propagating the (total) uncertainty embedded *in* the distribution. This type of total U management is essential for fuzzy numerical induction which aggregates lower fuzzy values into higher level fuzzy values. Rather than depending on an inferencing algorithm it is based on an understanding of all the uncertainty present in the fuzzy distributions. This type of U propagation has also been termed “ampliative reasoning” by Klir [59].



**Figure 1:** A Fuzzy Set or Possibility Distribution

Thus, a fuzzy set summarises the degree to which some elements belong to a finite set. This degree is usually termed the grade of membership ( $\mu$ ). In Figure 1, for example,  $L = 6.5$  belongs to the set of possible  $L$  values with 0.6 degree of membership. Or if  $L$  is the lifetime of a component, and one set element is  $x = 6.5$  months, then the degree of membership that  $L = 6.5$  months is 0.6 ( $\mu$ ) of total identity. However, a possibility measure adopts another interpretation of the same information, which is that 6.5 months = Lifetime, has a possibility or plausible belief level of 0.6 of maximum belief. So a possibility distribution represents a collection of set elements ordered by their individual belief values. But, more precisely, the distribution itself actually represents the *boundary* of a set of belief values. A key difference between the following analysis and that of normative Fuzzy Set and Possibility theory (which mainly focus on set-theoretic operations on the fuzzy distribution boundaries) is that this analysis focuses on the uncertainty forms embedded *inside* the possibility distribution. This shift in emphasis away from the boundary of a possibility distribution (or fuzzy set) then enables a different interpretation of the uncertainty content in such distributions.

As an example of the traditional interpretation, Klir and Yuan [58, p189] state that the possibility distribution function  $r(x)$ , uniquely defines the possibility measure of any subset  $A$  of a finite power set according to:

$$\text{pos}(A) = \max r(x) \quad \text{for } x \in A \quad \text{and} \quad r : X \rightarrow [0,1] \quad (5)$$

However, this defines unique possibility measures only when there is no ambiguity in the definition of the set elements themselves (and consequently in the subsets of elements). In other words, this definition implies that the set elements are point-valued, and consequently, the possibility distribution embodies only one dimension of uncertainty about the variable values. Then for point-valued variables, ambiguity meaning the divergence of possible values from a singleton, is represented by the set of elemental values and their corresponding belief values. As also explained by Klir and Yuan [58], multiple levels of uncertainty in these *belief degrees of variable values* can be recursively modelled by so-called “Type n” fuzzy sets. However, for practical purposes, these are usually limited to Type 2 sets, with an additional simplification of using only the support of the Type 2 fuzzy boundary itself to define a *band* or *interval* of uncertainty around the primary fuzzy distribution. But this Type 2 boundary form of vagueness is not sufficient to represent vagueness adequately. Rather, the other notion in the fuzzy set literature of Level 2 fuzzy sets [58] (not to be confused with the Level 2 modelling U of this author) is more appropriate, as will be subsequently explained.

A possibility distribution then represents an envelope of U regarding the possible existence of different values for set elements. Such possibility distributions are necessarily inexact by virtue of the high levels of U present, and any attempt to specify them exactly using parametric distributions estimated from data would be of doubtful value. In this way, a qualitative interpretation of possibility distributions is adopted in this report whereby simple forms such as triangular (TPD) or trapezoidal (TpPD) distributions are sufficient to capture belief variations over set element values. Nevertheless, the simple qualitative interpretation of a possibility distribution does enable the clear representation of the composite uncertainty forms that may be present.

### 3.4.2 Two Dimensions of Possibility

The forms of pre-computational U described in Section 3.3 individually, or in combination, induce two distinct forms of fuzziness which determine the basic class of the variable. These two forms of fuzziness can be described using the traditional fuzzy set representation with universe of discourse on the real ‘x’ axis and degree of possibility or belief ( $\pi$ ) on the ‘y’ axis. The two basic forms of fuzzy U correspond to two forms of possibility that may coexist. Thus a single possibility distribution is an envelope which can embed two distinct dimensions of possibilities within it. These two possibility dimensions are next described.

**Y-Possibility:** Represents a set of belief values for possible values of the variable’s elements across the universe of discourse. This is the standard form of possibility that relates to ambiguity as belief in multiple possible elemental values, which may be for point-value or interval-value elements.

**X-Possibility:** This form of U is induced when there is a band of *indistinguishable x values* possible at any level of belief because the variable intrinsically has interval-valued set elements due to vagueness. Thus the possibility distribution is an envelope over a nested set of fuzzy intervals. It is only the real-world U sources which are present that determine

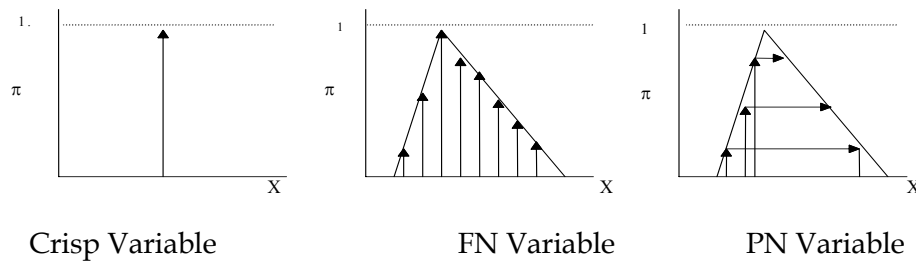
whether such a nest of fuzzy intervals exists. Although the concept of "E-Possibility" introduced by Dubois and Prade [28] for "Everywhere possibility" with multiple and indistinguishable values sounds very similar, X-possibility *is not always* present in contrast to E-possibility which is.

### 3.4.3 Three Types of Variables

The three types of variable that may be present in a system where there may be hybrid sources of uncertainty will now be described, and Figure 2 illustrates the different possibility dimensions that may be present in each of these three types of variable.

#### Crisp Variables

Crisp variables are quantitative and unambiguous variables, capable of accurate measurement and subject to stochastic variability. This is the non-fuzzy singleton type of variable as is treated by traditional statistical procedures. Some examples are the number of maintenance hours, surface temperature, or number of attacking aircraft. All of these may be represented by parametric statistical distributions.



**Figure 2:** Possibility Dimensions in Different Types of Variable

#### Fuzzy Numbers (FN)

A FN is a variable for which real point values can exist and be associated with varying belief values across their possible range (universe of discourse). FN can be induced by ambiguous evidence for crisply defined variables, which may be due to ambiguity within values (definition or measurement), or between values (contradictory or insufficient information), or both. Fuzzy variables may have varying degrees of U hybridisation and the concept of Uncertainty Dimensions (UD) will be introduced (and subsequently expanded) to describe the degree of uncertainty hybridisation or number of layers of distinct uncertainty in a variable. Excluding uncertainty due to statistical variations in sample evidence, for FN the basic or atomic UD = 1, representing the Y-possibility dimension.

$$\pi(x) = \text{Degree of possibility of a set element value } x$$

(maximum degree = 1, No possibility = 0)

Some triangular FN examples are: Measured size is "about 11" represented as a TPD (9,11,13), or "slightly less than 11" as TPD (9,10,11).

### Possibilistic Numbers (PN)

These variables have inherited X-possibility plus Y-possibility due to the epistemic interval nature of the variable elements caused by the U sources present. The difference between this type of epistemic interval-valued variable with a separate X-possibility form, and traditional fuzzy intervals will be explained in the next section. By viewing a PN variable as a set of consonant intervals they can be considered as Level 2 sets where the elements themselves are intervals. PN variables may arise from a qualitative definition or by subjective measurement. They may also arise from the approximate measurement of an approximately defined variable, or if estimation is based on a sample that is not representative of a universe of discourse. These potential sources of U and their combinations will be discussed in the following section but some examples of PN variables by definition are: Financial Risk or Military Threat level; and by measurement are: a guessed value of 8.5, or a projected inflation rate next year of 4.2%. It is important to identify the presence of PN in computations since the X-possibility can affect the choice of an operator in inductive computations.

#### 3.4.4 Fuzzy Intervals

Two distinct types of fuzzy interval will now be described: FN and PN intervals. Any interval variable requires two values to define the potential interval values. However the specific values required can differ between the two different types of interval variable. Using a triangular possibility distribution, Figure 3 illustrates the different possibilities inherent in these two different types of fuzzy intervals. Figure 4 then quantitatively demonstrates the different possibility variations for a triangular and a trapezoidal possibility distribution. Figure 4(b) also illustrates how X-Possibility may be evaluated in PN fuzzy sets.

Intervals For Point-Valued Variables:

( FN Intervals)

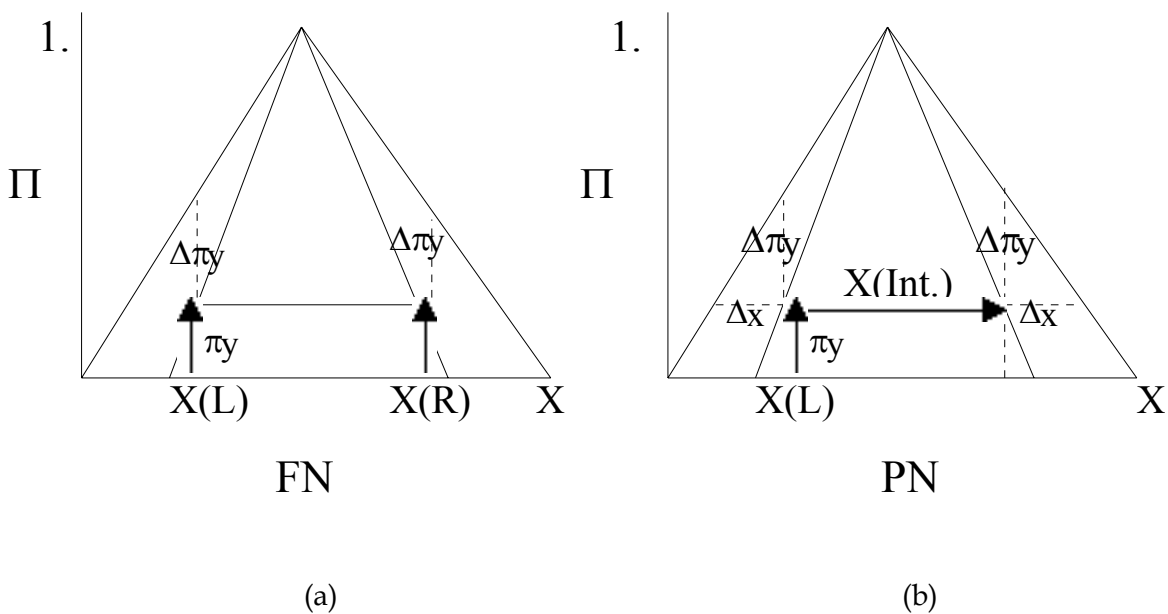
With this type of interval representation for a point-valued variable (Figures 3a and 4a) the extremes are *independent* ( X(L) and X(R) in Figure 3a). For this reason, only Y-possibility exists at each point-value and the possibility distribution represents only belief U estimates in the Y-possibility dimension ( $\Pi_Y$  = degree of belief in a value). And for the adopted qualitative interpretation of possibility distributions, an U margin is implicit at every plausibility degree ( $\Delta \Pi_Y$  in Figure 4a) since it cannot be exact.

Epistemic Interval-Valued Variables:

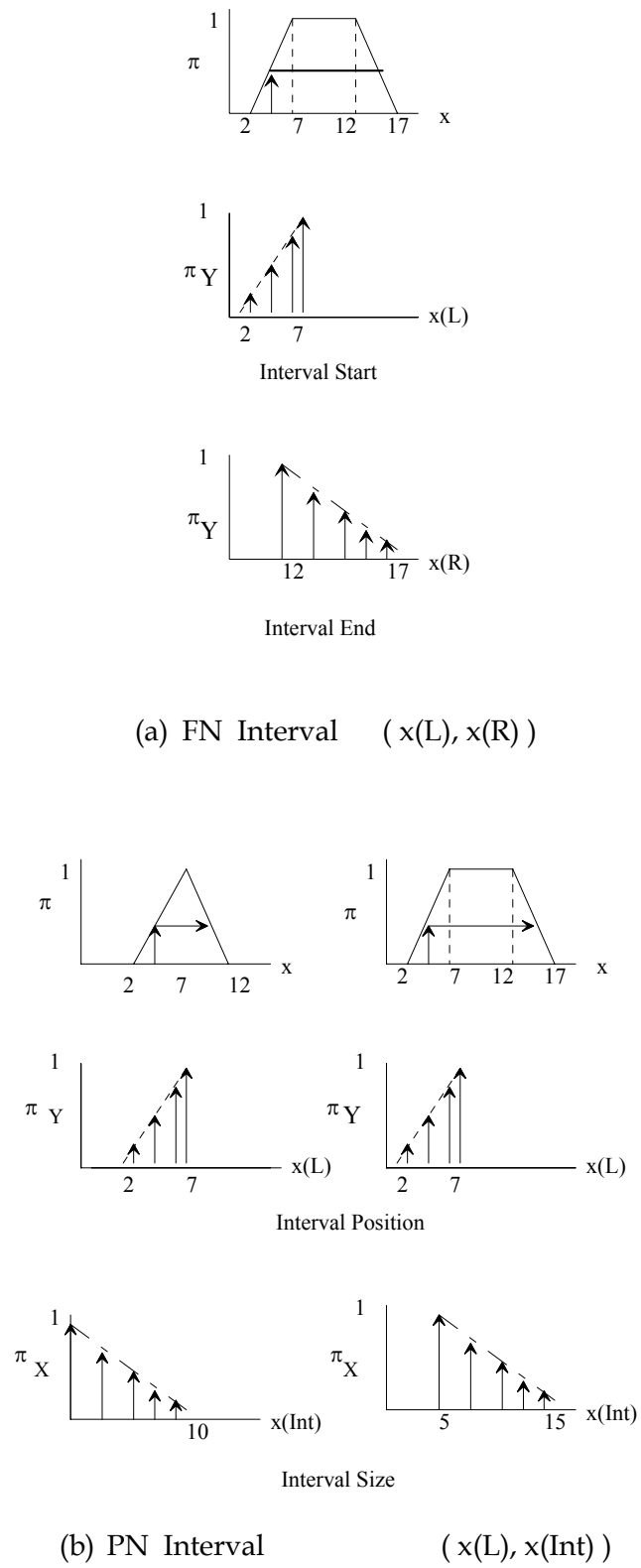
(PN Intervals)

For this type of interval-valued variable, the interval extremes are *not independent* because the epistemic level of vague U introduced by the U sources, induces discrete interval elements. The first value that identifies this interval at any degree of plausibility is the left extreme,  $X(L)$  in Figure 3b. Y-possibility at this value represents its degree of plausibility, with associated  $\Delta(\Pi_y)$  for the qualitative distribution understanding. Some similar but independent  $\Delta(\Pi_y)$  also exists at  $X(R)$ . Furthermore, for PN intervals an additional form or dimension of U exists at each point on the possibility distribution itself because the interval elements also embed epistemic belief U. This additional form of U ( $\Delta x$ ) in the *interval size* induces another separate and distinct dimension of possibility (X-Possibility), *which represents the feasibility that any interval size and all subsets of it exists*. Thus, at any point on a PN possibility distribution (on either side) *two* measures  $X(L)$  and Interval Size  $X(Int)$  are defined because the two extremes are connected and *not independent*. Accordingly, at any point on this distribution there are two possible forms of U associated with the measures for  $X(L)$  and  $X(Int)$ . Thus, the basic UD = 2.

Figure 3b illustrates the dual dimensions of U (and hence possibility) in PN and Figure 4b illustrates how *both* forms of possibility can be extracted from a single PN possibility distribution (TPD or TpPD). Figure 4a then illustrates the *single*  $\Pi_y$  dimension at the ends of a trapezoidal FN interval possibility distribution. The quantification of the fuzzy U forms associated with these two dimensions of possibility will be discussed in Section 4 after next discussing the various combinations of U that can induce the two types of fuzzy variable.



**Figure 3:** Two Types of Fuzzy Interval



**Figure 4:** Possibility Dimensions in Fuzzy Intervals

### 3.5 Hybrid Uncertainty in Variables

The various combinations of potential U sources will now be described and the resultant types of fuzzy variable that can be induced.

#### 3.5.1 Intrinsic Hybrids

As described in Section 3.2, intrinsic fuzziness can be induced by definition or measurement. Different hybrid combinations result in different dimensions of U. The possible combinations of intrinsic fuzziness are shown in Table 1 with the composite UD shown in each cell corner and induced variable type shown in each cell.

#### 3.5.2 Intrinsic Hybrids with Evidential Uncertainty

Evidential U may also be present in addition to the intrinsic fuzziness. In this case, another dimension of statistical uncertainty is inherent in a measure, usually quantified as probabilities. Furthermore, the information in the evidence may not be sufficient for rigorous statistical analysis adding further U to the probability estimates. Table 2 illustrates these evidential U situations with the possible UD again in cell corners. It is also postulated that when crisp information is statistically insufficient, the statistical estimators become fuzzy numbers (FN) due to their approximate estimation, for example, fuzzy means and fuzzy probabilities. In Table 2,  $P$  represents a crisp probability while  $\tilde{P}$  represents a fuzzy *likelihood* measure. For statistically sufficient information,  $P$  is a crisp probability estimate for both FN and PN type variables (X).

For statistically insufficient information, and where the sample space is representative of the universe of discourse, the fuzzy likelihood  $\tilde{P}$  is a FN ambiguous probability, again for both types of fuzzy variable ( $V = \text{FN or PN}$ ). However, when the sample space is not representative of the universe of discourse, the fuzzy likelihood  $\tilde{P}$  becomes a possibilistic number (PN). In this case  $\tilde{P}$  is not a strict probability and probability laws should not be applied because there is insufficient information in the measure.

These various forms of U hybridisation in evidence may also affect the method used to reduce a body of soft information to a measure of typicality. That topic will be discussed in Section 6 of this report.

#### 3.5.3 Uncertainty Dimensions of Hybrids

Table 3 summarises the UD that may potentially exist in a measure for various intrinsic and evidential fuzzy U combinations. In this classification scheme, the maximum UD for a crisp variable is 1 when sufficient information exists about statistical variations. Beyond that it must change to one form of a fuzzy quantity, which is determined by the U sources present. Table 3 also illustrates that the type of fuzzy variable induced is determined by the U forms present, rather than the UD degree (FN have a UD range of 1 to 3 while PN have a UD range of 2 to 6).



**Table 1: Intrinsic Uncertainty Hybrids**

			MEASUREMENT		
			QUANTITATIVE		QUALITATIVE
			<sup>0</sup> ACCURATE	<sup>1</sup> INACCURATE	<sup>2</sup> SUBJECTIVE
VARIABLE DEFINITION	QUANTITATIVE	<sup>0</sup> CRISP	0 C	1 FN	2 PN
		<sup>1</sup> APPROXIMATE	1 FN	2 PN	3 PN
	QUALITATIVE ( VAGUE )	<sup>2</sup> SEMANTIC	2 PN	3 PN	4 PN
		<sup>2</sup> CONTEXTUAL	2 PN	3 PN	4 PN
		<sup>2</sup> HYPOTHETICAL	2 PN	3 PN	4 PN

**Table 2: Evidential Uncertainty Hybrids**

		EVIDENTIAL UNCERTAINTY	
		1 STATISTICALLY SUFFICIENT	2 STATISTICALLY INSUFFICIENT
VARIABLE TYPE (V)	<sup>0</sup> CRISP	1 P	2 $\tilde{P}$
	1 FN	2 Pr(V is FN) is P	3 Pr(V is FN) is $\tilde{P}$
	<sup>2,3,4</sup> PN	3,4,5 Pr(V is PN) is P	4,5,6 Pr(V is PN) is $\tilde{P}$

**Table 3: Uncertainty Dimensions in Measures**

VARIABLE TYPE	Intrinsic Uncertainty Dimensions	Uncertainty Dimensions in a Sample of Evidence	
		Statistically Sufficient	Statistically Insufficient
CRISP	0	1	2
FUZZY NUMBER	1	2	3
POSSIBILISTIC NUMBER	2	3	4
	3	4	5
	4	5	6

### 3.5.4 Higher-Order Uncertainty Hybrids

The concept of higher-orders of U should be distinguished from the multiple number of U sources (and U Dimensions) as described previously. Strictly speaking, higher-order U refers to multiple compounds of U measures, but in this text, higher-order U will simply mean increasing amounts of U. With this understanding, some examples of higher-order U are shown in Table 4, with the two types of fuzzy quantity previously defined ranging across 2 to 4 orders of U, depending on the base variable. This report differentiates between ambiguity and vagueness to facilitate the identification of the third and fourth order U forms that combine the "what can eventuate" possibility measure, with the "what will eventuate" probability measure. It is not uncommon in strategic decision models to encounter higher-orders of U through the use of subjective likelihood estimates and other types of vague estimates. In Table 4 subjective probabilities, like objective probabilities, have been designated to be first order uncertainty because they are usually treated as quasi-objective probabilities by updating operations (as in Bayesian techniques). Thus, the meaning of such temporary estimates is actually that of a tentative "will be" estimate.

A fuzzy ambiguous scalar value (X), when it is not an uncertainty measure itself, is described as second order uncertainty because it represents an approximate estimate with varying levels of belief across *actual* values. Probability intervals, limits, and probability of probabilities, are also described as second order uncertainty since they indicate an additional degree of probabilistic multiplicity. There are also several other uncertainty formalisms [1,89] based on so-called "belief" measures which are special types of probabilities, however only the most popular Dempster-Shafer (D/S) measures have been included in Table 4. The D/S measures of Belief: representing the probability that evidence supports an event, and Plausibility: representing the probability that evidence does not exclude the event, is somewhat limited in the face of soft or sparse data because it requires a significant amount of data to estimate meaningful values. (Note that the D/S probabilistic definition of plausibility is different to the possibility meaning of plausibility described in Section 3.4.1). However, the D/S Plausibility measure has also been shown [24] to be a special case of the more general possibility measure.

Fuzzy ambiguous probabilities, as varying belief levels across multiple probabilities, have been described as third order uncertainty from the sum of their component orders. A fuzzy *vague* scalar variable (X) is also a third order of uncertainty as a varying belief distribution of *possible* (but not *actual*) values.

A *vague* probability then represents fourth order uncertainty in the framework of this report based on the possibility measure. The corresponding meaning is a "will be" that "can be" which closely approximates the meaning of what "may be". Thus, subjective probabilities, as vague probabilities, could be interpreted as fourth order U in the framework of this report, although when there are treated merely as tentative probabilities before updating with evidence they are simply first order U.

In relation to Table 4, it could be argued that fuzzy ambiguity of a scalar variable should only be first order uncertainty because there are no multiples of uncertainty scale metrics. While this is true for a strict definition of "order of uncertainty", the looser interpretation adopted

here simply means a less certain representation with more uncertainty. For soft information when third and fourth orders of U are present, many U modelling formalisms are severely limited when the quantification of vagueness is required for the robust management of U in numerical induction models. Although some definitions of vagueness have been proposed (as surveyed in Section 2) they are in general not suitable for propagating U in numerical induction models. Nevertheless, deductive fuzzy inferencing models as widely used in control applications are not greatly affected by vagueness, as they apply IF-THEN reasoning based on mapping fuzzy set membership values across fuzzy set boundaries.

### 3.6 Section Summary

This section has presented a framework for identifying uncertainty sources in information and associated each hybrid U combination with two distinct types of possibility that may be induced. On this basis, the forms of possibility induced by different hybrid U combinations have then been associated with two distinct types of fuzzy quantities: Fuzzy Numbers and Possibilistic Numbers. In these two types of fuzzy quantity the dominant U forms that are intrinsic are ambiguity and vagueness respectively. Thus, Possibilistic Numbers are a new type of fuzzy quantity which require that the variable necessarily be treated as a nested collection of fuzzy interval subsets wherein an additional non-standard form of possibility exists. Thus, two basic forms of fuzziness may be present in hybrid uncertainty systems, and the additional form of uncertainty in Possibilistic Numbers can affect the choice of algebraic operators in information synthesis. Consequently, when vagueness is present in information it should be explicitly modelled and carefully managed in induction models that hope to be robust.

**Table 4: Higher-Order Uncertainty Forms**

Order of Uncertainty	Example Uncertainty Representations	Meaning	Numerical Examples
Zero Order	A crisp value (singleton) of X	Actual value of X	7.3
First Order	A crisp Interval of X values	Range of equally possible values	[6.0, 9.0]
	Probability of (X) value/event (Objective)	Likelihood X Will Be	0.8
	Bayesian Probability of (X) (Subjective)	A Tentative Likelihood X Will Be	0.8
Second Order	Fuzzy ambiguous X (FN)	Fuzzy belief distribution (.) of <i>actual</i> X values	FN (TPD) = { 6.5(0.1) , 7.3(1); 8.8(0.1) }
	Crisp Interval of Probability of X	Rough X Will Be Likelihoods	[0.4, 0.8]
	Probability Limits of X event	Boundary Will Be Likelihoods	{0.4, 0.8}
	(D/S "Belief" & Plausibility) of X event	Belief in X (Will Be, Not NotBe)	{ 0.4, 0.5 }
	Probability of Probability of X event	Likelihood of the X Will Be Likelihood	{ 0.4, 0.8 }
Third Order	Fuzzy Vague X (PN)	Fuzzy belief distribution (.) of <i>possible</i> X values	PN (TPD) = { 5.0(0.1) , 7.8(1); 9.5(0.1) }
	Fuzzy Ambiguous Probability of X event	Fuzzy belief distribution of <i>actual</i> X values or event Likelihoods	FN (TPD) = { 0.4(0.1) , 0.6(1); 0.8(0.1) }
Fourth Order	Vague Probability (PN)	Fuzzy belief distribution (.) of <i>possible</i> X values or event Likelihoods  ( The Will Be that Can Be = The May Be )	PN (TPD) = { 0.2(0.1) , 0.8(1); 0.9(0.1) }

## 4. A New Approach to Fuzzy Uncertainty Measurement

### 4.1 Introduction

The new fuzzy uncertainty measures that will be proposed follow quite naturally from the descriptions of fuzzy ambiguity and vagueness presented previously. Although they are extremely simple, it is suggested that they present a more complete picture of fuzzy U than the traditional U measures. Furthermore, they are defined without the need for any non-fuzzy U concept, such as entropy, and this contributes to their integrity because such concepts cannot be assumed to be directly transferable to the soft information domain. Entropy implications for fuzzy U will also be explored later.

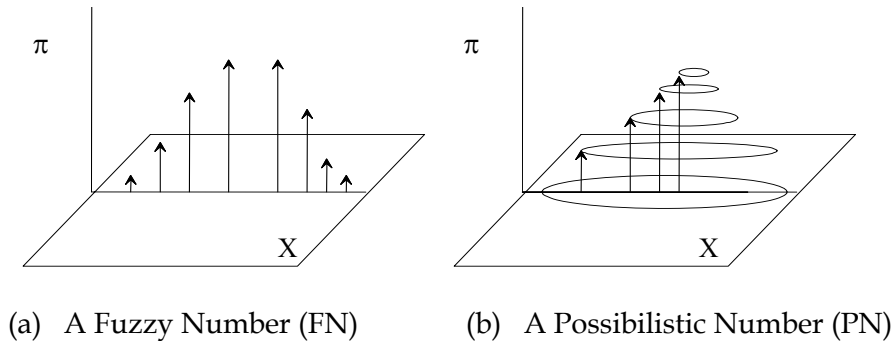
### 4.2 New Ambiguity and Vagueness Measures

The fuzziness that may exist in a fuzzy set representation may be of two types: “ambiguity”, associated with multiple possible elemental values, and “vagueness” characterised by indistinct interval elements. As previously discussed, it is the hybrid sources of U that epistemically determine what types of fuzziness are present. The traditional form of possibility (called here Y-possibility) measures the degree of belief in a set element's value, and consequently, can be used to measure the *multiplicity* of possible elemental values present in the set as follows:

$$\text{Ambiguity} \propto \int_0^{\infty} \text{pos}Y(x).dx \quad (6)$$

In addition, a new and non-standard form of possibility (called X-possibility) was identified as being associated with vague set elements. The aggregate of this X-possibility when it is present may then be quantified from the intervals ( $\alpha$ -cuts) in the X-possibility dimension as follows:

$$\begin{aligned} \text{Vagueness} &= \text{degree of indistinct “intervalness” of a fuzzy set} \\ &= \text{sum of intervals in X-possibility dimension} \\ &\propto \int_0^1 x \text{ interval}(\alpha).d\alpha \end{aligned} \quad (7)$$



**Figure 5:** Multiple Information Elements in Fuzzy Sets

Figure 5 illustrates these concepts and the two forms of possibility that may be present. The arrows in Figure 5a indicate that only one dimension of possibility is associated with elements in FN variables. The elliptical areas in Figure 5b represent the elements of a PN variable as a nest of indistinct intervals, and with a dimension of possibility associated with the size of an interval element, in addition to the possibility that a particular interval size exists (as in Figure 3). The arrows in Figure 5b position each interval element on the real axis, and the summation of those possibilities indicates the number, or multiplicity, of interval elements. Thus, there must also be the ambiguity form of U in a vague PN variable. Furthermore, variations in arrow size and interval element size are independent of each other (as shown in Figure 5b). The total fuzzy uncertainty in a PN is then the sum of both U forms present. However, rather than being absolute measures, these respective summations possess only relative meaning in relation to a “typical” value of the set of elements. For simplicity, the typical value for both FN and PN variables will be taken as the Centre of Area (COA) of the fuzzy distribution. This point represents a “median” value where the total possibility of being less than, equals the total possibility of being greater than the COA. (It should also be noted that the commonly used Centroid cannot be interpreted as an average because the product of a real value by its possibility is meaningless.) However, a relative U measure as a ratio of the U content over the COA, may be undefined when the COA = 0, which could be the case when modelling measurement error for example. In these cases, the numerator U area only need be used since the purpose of defining such a proportional U measure is merely to add some perspective to the metric by comparing the U content to the *order* of the fuzzy quantity. To determine the ambiguity U content in PN, the left-hand value of the most plausible interval (which may be a zero or even a discontinuous interval) at Sup ( $\pi$ ) is taken to be the real value *which bounds the multiplicity of intervals*. As a first approximation, the area left of this value then determines the Ambiguity measure for the PN. However, a more accurate method to determine the ambiguity content within a PN will be introduced in the next section. To determine the amount of vagueness in a PN, the area of the whole distribution measures the total “intervalness” of the fuzzy set. From these notions measures for ambiguity and vagueness can now be summarised.

Fuzzy Numbers (FN):

Vagueness = 0

Ambiguity = Total Area of Fuzzy Set / Centre of Area

Possibilistic Numbers (PN):

Vagueness = Total Area of Fuzzy Set / Centre of Area

Ambiguity = Area Left of Sup ( $\pi$ ) / Centre of Area (Approx.)

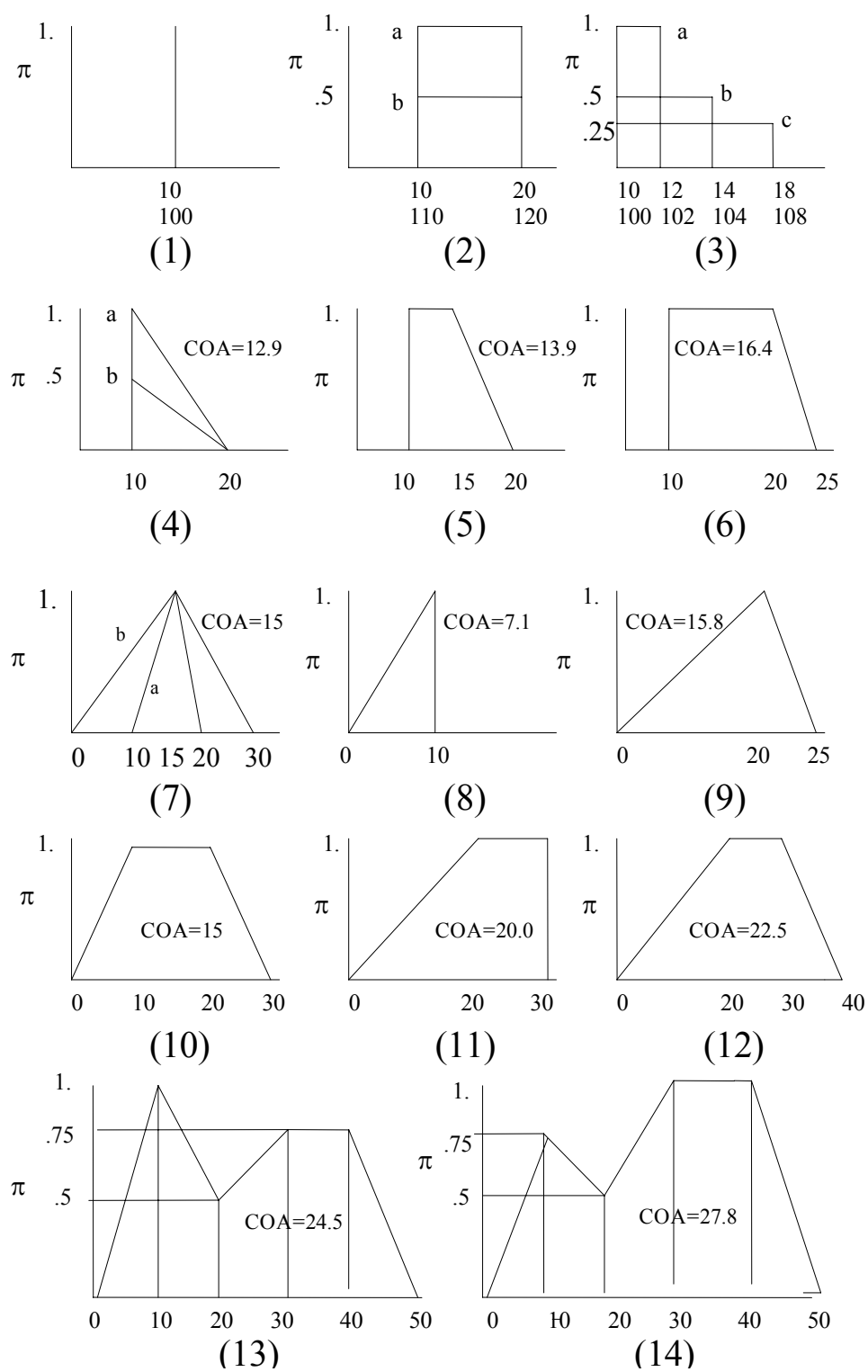
Figure 6 illustrates a variety of linear possibility distributions to demonstrate how different degrees of ambiguity and vagueness may be present. Table 5 summarises the proposed Ambiguity and Vagueness measures for both FN and PN interpretations of the example linear distributions in Figure 6, when the fuzzy quantities are of order 10 and also of order 100.

### 4.3 Ambiguity Measurement in Vague Variables

The method described previously to quantify the *ambiguity* uncertainty content within a vague PN type variable was only a first approximation. It has been introduced only for simplicity to demonstrate the behaviour and rationale of the new fuzzy U measures proposed. Strictly speaking, the Ambiguity in a PN, as the number of intervals present, can only be quantified by the above method if the fuzzy set is symmetrical. If the fuzzy set is not symmetrical, one could ask why the left side is used to evaluate ambiguity rather than the right side. As it turns out, both sides can be used to form an average ambiguity evaluation for a PN by means of the traditional Measure of Fuzziness, being the intersection of a set and its complement. This is applicable even for irregular shaped fuzzy distributions because it is a function of all the different gradients present in a fuzzy set, which reduces to the approximation above in the case of symmetrical fuzzy distributions. The general behaviour of the Measure of Fuzziness will be described in more detail in Section 4.4 which demonstrates the measure's behaviour over a variety of fuzzy sets. However, it should be noted that the Measure of Fuzziness does not reflect the ambiguity in FN type fuzzy variables which should be quantified as previously described. So this well-established measure of fuzziness can be given a new and explicit meaning in this different approach to fuzzy U analysis, whereby it *represents the ambiguity content of a vague PN variable*. Notably, this meaning is quite the opposite of the most prevalent understanding in the literature, which is that it measures vagueness due to concept graduality. Pal [77] has also stated that the Measure of Fuzziness represents "the average ambiguity in a fuzzy set". Pal considers fuzziness to be one of three major forms of U, the others being probabilistic U and Nonspecificity, as "the degree of concentration of elemental mass assignments". Thus, while Pal's concept of ambiguity is similar to that of this report for ambiguity in a PN, it does not equate exactly to the precise definition of general element multiplicity used here.

### 4.4 Some Fuzzy Uncertainty Measure Comparisons

The behaviour of the proposed ambiguity measure (Ambiguity) will now be compared with that of the traditional measures of Specificity and U-Uncertainty using a variety of linear possibility distributions (Figures 7, 8, 9). The normalised version of U-Uncertainty (as recommended by Klir [57]) and a modified Measure of Fuzziness (as previously defined) is also included in the summarised results in Table 6. Since the traditional measures cannot distinguish vagueness, all the examples in Figures 7, 8, and 9 can be considered to be FN, as fuzzy quantities where only ambiguity U is present. Figure 7 actually depicts crisp intervals but for comparison purposes they will be considered as possibility distributions where belief is restricted to  $\{0,1\}$  with no partial values. The distributions in Figure 8 are also crisp possibility distributions, with partial but no graded belief, while those in Figure 9 are fuzzy possibility distributions with graded belief. Specificity is measured as an index  $[0,1]$  with Non-Specificity (NSP) =  $1 - \text{Specificity}$ . Two units for U-Uncertainty are included: bits (log base 2) and digits (log base 10). It should be noted that with the U-Uncertainty measure for TPD there is no direct conversion between different units because of the subtraction of a constant in the basic formula (generalised U-Uncertainty and Specificity formulas for TPD will be developed in Section 5). The units for the Measure of Fuzziness are the number of elements of the support set. Table 6



**Figure 6:** Example Linear Possibility Distributions



**Table 5:** Values of Proposed Uncertainty Measures for Figure 6 Distributions

FIGURE 6	FUZZY NUMBER		POSSIBILISTIC NUMBER			
	AMBIGUITY		VAGUENESS		AMBIGUITY	
	Order10	Order100	Order10	Order100	Order10	Order100
1	0	0	0	0	0	0
2a	66% (10/15)	9% (10/115)	66% (10/15)	9% (10/115)	0	0
2b	33% (5/15)	4% (5/115)	33% (5/15)	4% (5/115)		
3a	18% (2/11)	2% (2/101)	18% (2/11)	2% (2/111)	0	0
3b	17% (2/12)	2% (2/102)	17% (2/12)	2% (2/112)		
3c	14% (2/14)	2% (2/104)	14% (2/14)	2% (2/114)		
4a	39% (5/12.9)	4% (5/112.9)	39% (5/12.9)	4% (5/112.9)	0	0
4b	19% (2.5/12.9)	2% (2.5/112.9)	19% (2.5/12.9)	2% (2.5/112.9)		
5	54% (7.5/13.9)	7% (7.5/113.9)	54% (7.5/13.9)	7% (7.5/113.9)	0	0
6	77% (12.5/16.4)	11% (12.5/116.4)	77% (12.5/16.4)	11% (12.5/116.4)	0	0
7a	33% (5/15)	4% (5/115)	33% (5/15)	4% (5/115)	17% (2.5/15)	2% (2.5/115)
7b	100% (15/15)	13% (15/115)	100% (15/15)	13% (15/115)	50% (7.5/15)	7% (7.5/115)
8	71% (5/7.1)	5% (5/107.1)	71% (5/7.1)	5% (5/107.1)	71% (5/7.1)	5% (5/107.1)
9	79% (12.5/15.8)	11% (12.5/115.8)	79% (12.5/15.8)	11% (12.5/115.8)	63% (10/15.8)	9% (10/115.8)
10	133% (20/15)	17% (20/115)	133% (20/15)	17% (20/115)	33% (5/15)	4% (5/115)
11	100% (20/20)	17% (20/120)	100% (20/20)	17% (20/120)	50% (10/20)	8% (10/120)
12	111% (25/22.5)	20% (25/122.5)	111% (25/22.5)	20% (25/122.5)	44% (10/22.5)	8% (10/122.5)
13	124% (30.3/24.5)	24% (30.3/124.5)	124% (30.3/24.5)	24% (30.3/124.5)	20% (5/24.5)	4% (5/124.5)
14	117% (32.5/27.8)	25% (32.5/127.8)	117% (32.5/27.8)	25% (32.5/127.8)	63% (17.5/27.8)	14% (17.5/127.8)

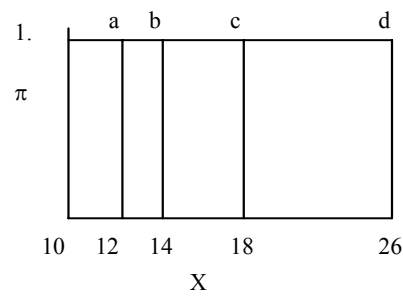
summarises the results of the comparative calculations for all the test sets and the calculation details are listed in Table 7. Some U-Uncertainty computations for similar TPD can also be found in Klir and Yuan [58, pp. 252].

Several observations can now be made based on comparisons of the results shown in Table 6 for the different measures (remembering that no vagueness is present in these examples).

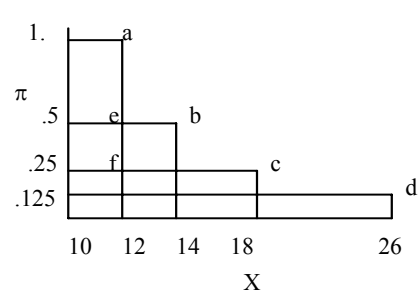
Figure 7 illustrates quantities with linearly increasing ambiguity and all measures are fairly well behaved for these binary belief distributions. Figure 8 introduces partial belief and composite variations in belief and ambiguity for crisp possibility distributions. The most exceptional deviations can be observed in the Specificity drop between *a* and *c* in Figure 8. Figure 9 also illustrates the effect of composite ambiguity and belief variations, in this case for fuzzy possibility distributions. For this group, an anomaly can also be seen in Specificity for *d* in Figure 9, which is about the same as that for *a* in Figure 9 (0.16 : 0.19) although the possibility distributions are *quite different*. This does not occur with the other two measures which reflect a decrease in ambiguity (from 9a to 9d). Overall, Specificity does not appear to be well behaved. While U-Uncertainty is more consistent, Figures 8 and 9 illustrate its insensitivity when dividing by the height ( $\mu$  Max), which reduces the measure to a function of the support only (Figures 8 a,e,f and 9 a,b and 9 c,d about equal within groups). Figure 9 also illustrates how the value of U-Uncertainty varies over a large range for log base 10, while the U-Uncertainty bit values for Figures 9b and 9d are about the same. Based on these simple possibility distributions, the results in Table 6 appear to demonstrate the intuitively reasonable and more consistent behaviour of the Ambiguity measure, as a measure of information on ambiguity.

The Measure of Fuzziness values can be simply determined by subtracting the sum of the Hamming distances from the support unit interval. The measure is zero for uniform crisp possibility distributions (Figure 7), but takes values for crisp distributions with partial belief values (Figure 8). Figure 9 demonstrates the *inability* of the unmodified version to discriminate between normal and non-normal possibility distributions. In comparison, the modified Measure of Fuzziness ranks the fuzziness of the Figure 9 distributions in the order  $b > a > d > c$ , which seems to be intuitively correct to this author. These simple examples show the effect of support size and maximum belief level on the Measure of Fuzziness values. For computations that utilise the Measure of Fuzziness, as does clustering for example, the most suitable definition should ultimately be determined by the context, and it is not implied that the modified version is always superior.

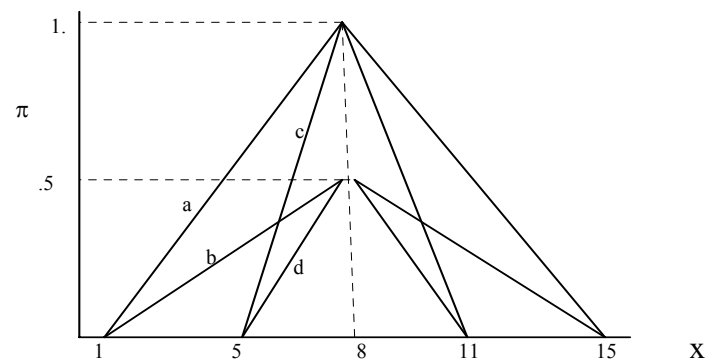
Overall, a major weakness of Specificity, U-Uncertainty, and the Measure of Fuzziness, is that they are independent of the order of magnitude of the fuzzy quantity, being solely determined by the distribution and spread. In contrast, the Ambiguity measure describes the amount of fuzzy information available in relation to the magnitude of the fuzzy quantity. This extra meaning enables estimates of different types of fuzzy U to be combined (as will be demonstrated in Section 6) and the clearer interpretation generally provides a semantic advantage in numerical induction.



**Figure 7:** Crisp Binary  
Possibility Distributions



**Figure 8:** Crisp Non-Binary  
Possibility Distributions



**Figure 9:** Fuzzy Possibility Distributions

**Table 6:** Comparison of Ambiguity Measures and Fuzziness Measures

Figure	Specificity [0,1]		U-Uncertainty			Ambiguity		Fuzziness	
	SP	NSP	Digits	Bits	Bits / $\mu$ Max	Info. %	Ratio	Trad.	Mod.
7 a	0.33	0.67	0.48	1.6	1.6	18	2/11	0	0
b	0.20	0.80	0.70	2.4	2.4	33	4/12	0	0
c	0.11	0.89	0.95	3.2	3.2	57	8/14	0	0
d	0.06	0.94	1.23	4.1	4.1	89	16/18	0	0
8 a	0.33	0.67	0.48	1.6	1.6	18	2/11	0	0
b	0.10	0.90	0.35	1.2	2.4	17	2/12	4	2
c	0.03	0.97	0.24	0.8	3.2	14	2/14	4	6
d	0.007	0.993	0.15	0.5	4.1	11	2/18	4	14
e	0.17	0.83	0.24	0.8	1.6	9	1/11	2	1
f	0.08	0.92	0.12	0.4	1.6	4.5	0.5/11	1	1.5
9 a	0.19	0.81	0.26	3.2	3.2	87	7/8	7.0	7.0
b	0.10	0.90	0.13	1.6	3.2	44	3.5/8	7.0	10.5
c	0.32	0.68	-0.014	2.3	2.3	38	3/8	3.0	3.0
d	0.16	0.84	-0.007	1.2	2.4	19	1.5/8	3.0	4.5

**Table 7:**        *Computations for Test Examples*

Figure	Specificity	U-Uncertainty		Measure of Fuzziness (By area subtraction)	
		Digits (log <sub>10</sub> )	Bits (log <sub>2</sub> )	Traditional	Modified
7 a	1/3	log 3 = 0.48	log <sub>2</sub> 3 = 1.6	2-2 = 0	2-2 = 0
b	1/5	log 5 = 0.70	log <sub>2</sub> 5 = 2.4	4-4 = 0	4-4 = 0
c	1/9	log 9 = 0.95	log <sub>2</sub> 9 = 3.2	8-8 = 0	8-8 = 0
d	1/17	log 17 = 1.23	log <sub>2</sub> 17 = 4.1	16-16 = 0	16-16 = 0
8 a	1/3 = 0.33	log 3 = 0.48	log <sub>2</sub> 3 = 1.6	2-2 = 0	2-2 = 0
b	1/5(1/2) = 0.10	1/2 log 5 = 0.35	1/2 log <sub>2</sub> 5 = 1.2	4-0 = 4	4-4(.5) = 2
c	1/9(1/4) = 0.03	1/4 log 9 = 0.24	1/4 log <sub>2</sub> 9 = 0.8	8-8(.5) = 4	8-8(.25) = 6
d	1/17(1/8) = 0.007	1/8 log 17 = 0.15	1/8 log <sub>2</sub> 17 = 0.5	16-16(.75) = 4	16-16(.125) = 14
e	1/3(1/2) = 0.17	1/2 log 3 = 0.24	1/2 log <sub>2</sub> 3 = 0.8	2-0 = 2	2-2(.5) = 1
f	1/3(1/4) = 0.08	1/4 log 3 = 0.12	1/4 log <sub>2</sub> 3 = 0.4	2-2(.5) = 1	2-2(.25) = 1.5
9 a	1/14 ln 15 = 0.19	15/14 log <sub>10</sub> 15 -1.0 = 0.26	15/14 log <sub>2</sub> 15-1.0 = 3.2	14-7 = 7	14-7 = 7
b	1/28 ln 15 = 0.10	15/28 log <sub>10</sub> 15 -0.5 = 0.13	15/28 log <sub>2</sub> 15 -0.5 = 1.6	14-7 = 7	14-(14/4) = 10.5
c	1/6 ln 7 = 0.32	7/6 log <sub>10</sub> 7 -1.0 = - 0.014	7/6 log <sub>2</sub> 7 -1.0 = 2.3	6-3 = 3	6-3 = 3
d	1/12 ln 7 = 0.16	7/12 log <sub>10</sub> 7 -0.5 = - 0.007	7/12 log <sub>2</sub> 7 -0.5 = 1.2	6-3 = 3	6-3(.5) = 4.5

## 4.5 Section Summary

Simple measures have been proposed for two distinct forms of fuzzy U, ambiguity and vagueness, that may be induced in a fuzzy quantity by various U sources. There seems to exist in the literature a general consensus that ambiguity concerns multiple possible meanings or values, meaning the degree that more than one attribute or value can possibly exist. And although vagueness is generally described as meaning “unclear” or “indistinct”, at present there is no clear definition in the literature for vagueness. The most dominant conception equates vagueness with the “fuzziness” form of U, as quantified by Measures of Fuzziness. On the basis of a careful analysis of the various *sources of fuzziness* and their meaning in terms of possibility theory, this conception of vagueness is considered to be inadequate because it does not capture higher-order U forms which induce indistinct set elements. Another understanding of vagueness frequently encountered in the literature is to equate it with Type 2 fuzzy sets with indistinct distribution boundaries. Although this report also adopts a qualitative interpretation of fuzzy sets, whereby possibility distributions always have imprecise boundaries, a key difference is that vagueness is here associated with the interval-nature of set elements rather than with the size of the band of possibility U around the belief distribution. In fact, with the approach presented here, it is not necessary to quantify the size of the band in order to capture the effect of vagueness on computations for numerical induction. However, for deductive type inferencing computations, it may make more sense to quantify the boundary band since it can affect the syllogistic mechanisms.

Ambiguity has been defined here as the degree of multiplicity of set elements and quantified using the traditional form of possibility which estimates the feasibility of each element's value as the degree of membership in the set. Vagueness has been defined as the aggregate degree to which the set possesses interval-valued elements; and the independent form or dimension of possibility associated with an interval's existence has been termed X-Possibility. The U measures proposed are intended to measure the proportion of the fuzzy quantity covered by the different types of fuzzy U present. This type of *relative* knowledge is absent in such traditional measures as U-Uncertainty and Specificity. In addition to this limitation, these two traditional measures have also been shown to be rather inconsistent and erratic, for some simple types of possibility distributions. The U-Uncertainty results have also demonstrated the sensitivity dependency of that measure on the log base used. It will also be shown in Section 5.3 that the log base is not merely an arbitrary choice of convenience, but is determined by the units or degrees of belief modelled (i.e. the belief quantisation). Although log base 10 may be a pragmatic choice, for continuous distributions it could be infinite and higher values mean higher sensitivity. Overall, it seems difficult to justify the demonstrated non-linear sensitivity of the U-Uncertainty measure, even when using log base 10. Consequently, the information on uncertainty that measure provides would seem to be dubious. Furthermore, these comments also apply to all measures that transpose directly the *probabilistic* entropy concept to a fuzzy distribution. For these reasons, it is suggested that the proposed measures may be more reliable and functional since the purpose of U measures is to ensure meaningful U representation for careful U *propagation* through computations. However, the primary reason for defining these new measures is that there is no traditional measure that captures the vagueness form of fuzzy uncertainty.

## 5. Extensions to Some Fuzzy Uncertainty Measures

### 5.1 Introduction

Extensions to some of the common fuzzy U measures encountered in the literature will now be presented. Derivations will be from first principles highlighting the meaning of such features as the base of the logarithmic functions. The purpose of these extensions is to highlight some assumptions implicit in these measures, and to show their limitations for higher-order U evaluation.

### 5.2 A New Fuzzy Entropy Measure

Stochastic variability may also be present in a body of soft evidence. Theoretically, this form of uncertainty can be quantified independently to the previous forms of fuzzy uncertainty using the well-founded measure of entropy based on an aleatory probability distribution as relative frequencies of occurrence within the sample. Uncertainty may be introduced by process variability that causes multiple possible values to occur. Separate and in addition to this there may be belief U, that may itself be comprised of an U component caused by randomness plus another component caused by different sources of fuzziness. Although the two types of stochastic U can theoretically be quantified using the entropy function as below, a very large data set may be required to derive meaningful probability distributions, especially for the probability of the value possibility estimate  $P'(\pi(x_i))$ . Whether to include the probabilistic U due to belief variations at all, even if a large sample were available, depends on answers to the following questions:

Are belief variations only due to imperfect observations, or are they caused by some randomness in the observation process, but not originating in the external world?

Observation randomness could occur if observations are reported at fixed intervals (say daily), but each observer (or expert) receives an amount of sensor data varying randomly due to the characteristics of the respective vantage points. These variations are not then caused by imprecision of observation, nor by variations in the observed entity or process, but by variations in sensor supplied information. Identifying the sources of U influencing a data set is essential for an adequate *interpretation* of the data and its *quantitative representation*, and these issues will be discussed further in Section 6.5, which presents a method for quantitatively representing total uncertainty. It should be noted that the following application of the entropy concept to soft information is quite distinct to that of the pioneering work of De Luca and Termini [20] in this area. Those authors directly applied the probabilistic entropy concept to fuzzy sets utilising the feature that the fuzzy set for *not* belonging to a set of real values, can be derived from the fuzzy set of belonging to a set of real values by simple membership complementation ( $1-\mu$ ). Their idea is that a fuzzy set embeds information on what *is not*, as well as information on what *is*, and the sum of these is the total information in the fuzzy set. They investigated axiomatic requirements for a function to evaluate these two entropies but did not actually define such a function, saying instead that there are many open questions. However, later authors have interpreted the fuzzy entropy concept of DeLuca and Termini by simply replacing the probability measure with the fuzzy U metrics  $\mu$  or  $\pi$ . For

example, Pal and Bezdek [76] present the following function for DeLuca and Termini fuzzy entropy  $H(A)$ .

$$H(A) = -K \sum_{i=1}^N [\mu_i \log \mu_i + (1-\mu_i) \log(1-\mu_i)] \quad (8)$$

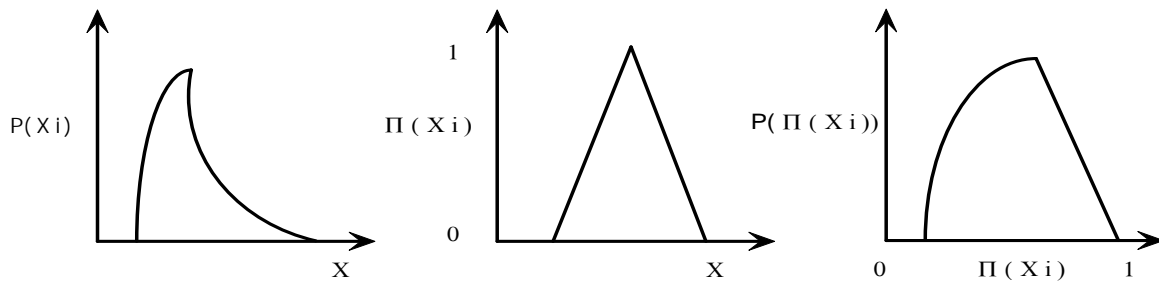
where  $K$  is a normalising constant.

Many other variants exist but unfortunately all such substitutions of fuzzy belief metrics for probabilities are meaningless because the probabilistic entropy function is based on the relative frequency concept. Thus,  $\mu \log \mu$  is a meaningless expression (discussed further in Section 5.3). The fundamental implications of the probabilistic information entropy construct are:

1. the variable must be a binary state variable (is or is not a value)
2. the entropy shows the expected information provided by an occurrence (or value) based on the number of possible binary  $\{0,1\}$  occurrences and their probability distribution.
3. the amount of expected information is the dual of the uncertainty due to probable variations.

Various entropy extensions, such as conditional entropy, cross entropy and mutual information follow directly from the basic relationship shown in this report (Section 5.3) regarding probable variations. Anderson [2] is one of the few authors to highlight clearly the meaning of Shannon entropy and it is unfortunate that many have been introduced to entropy by *fait accompli* presentations in the manner of: “it has been shown that the entropy function is the only function that satisfies a set of necessary axiomatic requirements....”. This illustrates certain limitations of axiomatic reasoning due to undefined behavioural gaps between basic, extreme, or boundary characteristics referenced in axioms. Many other authors [4][8][77][95] have investigated the application of Shannon entropy to the fuzzy domain but also from the viewpoint of axiomatic requirements, and with the implicit assumption of binary variables as inherent in the probabilistic entropy concept.

Thus probabilistic variability can be present in *both* the state values ( $x$ ) and their belief values ( $\pi$ ), and the total information on the variability present is the sum (entropy being additive) of the respective entropies of these two forms of variation (variable value and value possibility) as defined by the two probability distributions  $P(x_i)$  and  $P(\pi(x_i))$  based on multiple estimates (Figure 10).



**Figure 10:** Probabilistic Variations in a Fuzzy Quantity from Multiple Estimates

Let  $n$  = Number of fuzzy estimates (or experts),  $j = 1$  to  $n$   
 $m$  = Constant support cardinality of each fuzzy estimate  
 $x_i$  = Variable state values,  $i = 1$  to  $m$

*Variations of Value Possibility Estimates  $\pi(x_i)$  :*

Probability of  $\pi(x_i)$ :  $P' = \text{No. estimates } \pi(x_i) / \text{Total No. } (n)$   
 and Possibility variations information:  $H = \sum_{\pi=0}^1 P'(\pi(x_i)) \log_2 P'(\pi(x_i))$

So information across universe of all possibilities of all quantity values:

$$H = \sum_{i=1}^m \sum_{\pi=0}^1 P'(\pi(x_i)) \log_2 P'(\pi(x_i)) \quad (9)$$

$$= N1 \text{ Bits}$$

Since  $\pi(x_i)$  is a binary variable (e.g. equal 0.3 or not equal 0.3), the log base is 2.

*Variations of Value Estimates  $(x_i)$  :*

$$\text{Probability of } x_i = P(x_i) = \frac{\sum_{j=1}^n \pi(x_i)}{n}$$

$$\text{Total Sample Information (for all } m \text{ values)} = \sum_{i=1}^m P(x_i) \log_{10} P(x_i) \quad (10)$$

$$= N2 \text{ Decimal Digits}$$

Since  $x_i$  is not a binary variable, the Log base is quantisation of belief, say 10. However,  $N1$  and  $N2$  cannot be summed directly since the first is in Bits while the second is in decimal digits. Transposing  $N2$  to Bits gives :

For information equivalence:

$$\left(\frac{1}{2}\right)^{N2/} = \left(\frac{1}{10}\right)^{N2}$$

$$N2/ = \log_2 10^{N2} \text{ Bits}$$

$$= N2 \log_2 10 \text{ Bits} \quad (11)$$

Thus, total information ( $H^*$ ) on *both* quantity value variations and quantity value belief variations is  $N1 + N2/$  bits :

$$H^* = \sum_{i=1}^m \sum_{\pi=0}^1 P'(\pi(x_i)) \log_2 P'(\pi(x_i)) + N2 \log_2 10 \text{ Bits}$$

$$= \sum_{i=1}^m \sum_{\pi=0}^1 P'(\pi(x_i)) \log_2 P'(\pi(x_i)) + \sum_{i=1}^m P(x_i) \log_{10} P(x_i) \log_2 10 \text{ Bits} \quad (12)$$

As previously stated, a large data set may be required for this procedure since the above entropies can only be calculated if there is statistically sufficient information to calculate the crisp probabilities of value possibilities. This requirement may be hard to satisfy with imperfect information sets.



Some comparisons will now be made between the concepts presented in this section and the common usage in the literature of the terms: random fuzzy variable and fuzzy random variable. A random variable traditionally means a crisp variable subject to random fluctuations, so a fuzzy random variable means that only fuzzified forms of variational information is available due to the incomplete nature of the evidence. On the other hand, a fuzzy state variable is also represented by partial membership levels, so a random fuzzy variable refers to a variable estimated from a body of evidence containing information on the variations of the fuzzy states. However, for the purpose of quantifying the uncertainty in a body of soft evidence, this dichotomous probability/fuzzy division of U is not considered to be adequate by the present author, since all the potential hybrid combinations of uncertainty cannot be identified and distinguished. If the different forms of fuzzy uncertainty cannot be distinguished, the quantification of fuzzy uncertainty cannot be concise. An example of this problem can be noted in Cai's [12] method of possibilistic reliability analysis (to be analysed in Section 7). In Cai's approach to possibilistic reliability analysis, it is not clearly defined what possibilistic reliability really means and how it can be induced. As a consequence, an appropriate method of implementing logical operations for system reliability synthesis cannot be determined. Although a number of authors have investigated fuzzy random variables, (for example Kwakernaak [65], Puri and Ralescu [78] and Fukuda [39]), the use of only two U forms (probability and possibility) for modelling higher-order U variants does not provide sufficient discriminatory power to address all the actual U hybrids that can exist in information. Some extensions to the classical measures of fuzzy U will now be introduced.

### 5.3 Derivation of Shannon Entropy Law by Induction

One of the earliest U measures in 1928 was the Hartley Information measure [44] for discrete states. That measure was initially defined using a base 10 logarithm for decimal digits. Later developments in information theory have focused on base 2 representations as are more relevant to binary computers, and Shannon's paper [85] may be considered as the seminal reference in this regard. The fundamental notion of the Hartley information concept is how many digits are required to fully identify a particular state within the set of possible states or alternative values. These logarithmic information measures will be extended beyond crisp singletons (traditional versions) to crisp interval variables, and then to fuzzy variables, fuzzifying first the evidence and then the event states. Crisp belief measures produce a binary event space, while partial belief measures allow a non-binary event space. The following results also illustrate the effect of fuzzy events on the foundations of the U-Uncertainty measure (to be discussed further in Section 5.5).

*Crisp Singletons:* (Binary Belief, Crisp State)

In Figure 11, three binary digits can identify eight crisp states or possible values, described in the following basic relationship (13). The number of binary digits (bits) required to identify the state within its universe of discourse is the Hartley Information concept, where more bits are required for a less specific or larger universe  $|A|$ . For binary belief of an element's state (0 or 1) the following expression defines the number of bits required to identify a particular state from all possible states:

$$\left(\frac{1}{2}\right)^N = \frac{1}{|A|} \quad (13)$$

where:  $N$  = Number of bits to identify value,  $|A|$  = Number of possible values.

The fraction in the left side of (13) represents one of two possible states encoded at each binary digit, and the right side represents the choice of one from all the possible alternate values.

Then  $2^N = |A|$  and  $N = \log_2 |A|$  (Hartley Information)

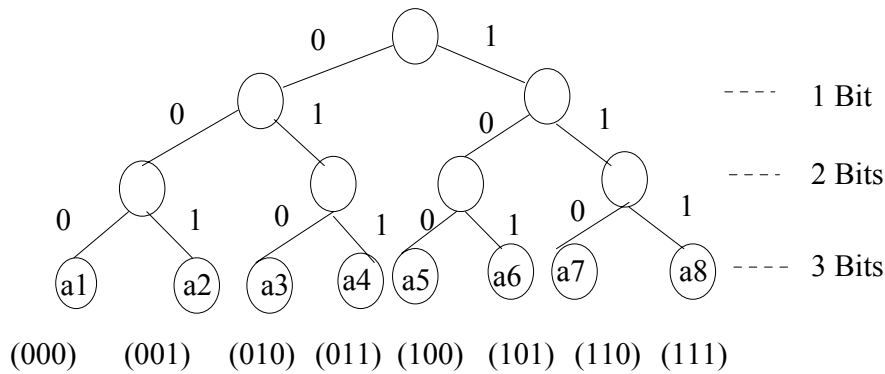
Consider a uniform probability distribution across all states:  $p = \frac{1}{|A|}$

$$\text{so } \left(\frac{1}{2}\right)^N = P \quad \text{and by definition} \quad N = -\log_2 p \quad (14)$$

This expression (14) is sometimes called Wiener Entropy. Then for statistical expectation, the *expected* information ("entropy") in a set of state/probability information is:

$$I = - \sum_{i=1}^{|A|} p(x_i) \log_2 p(x_i) \quad (15)$$

This is a direct inductive derivation of the Shannon entropy law [84].



**Figure 11:** A Three Bit Binary Event Space

*Crisp Interval Variable:*

(Binary Belief, Crisp State)

A crisp interval-valued variable requires two values to specify it and each of these must be totally one value or not (binary state). Equation (13) must then be modified to reflect the dual values required within the possible universe of discourse:

$$\text{Then} \quad \left(\frac{1}{2}\right)^N = \frac{2}{|A|} \quad (16)$$

where  $|A|$  is the Number of possible values.

$$\text{and} \quad 2^N = \frac{|A|}{2} \quad \text{or} \quad N = \log_2 |A| - \log_2 2$$

$$\text{so} \quad N = \log_2 |A| - 1 \quad (17)$$

Thus for a given number of states, Hartley Information is less for a crisp interval-valued variable than for a crisp point-valued variable.

*Fuzzy Interval Variable (FN):*

*(Partial Belief, Crisp State)*

In this case, partial belief values exist for being in any crisp state. The state space in Figure 11 is then modified according the degree of belief quantisation. If, for example, we allow 10 equal levels of belief in  $[0,1]$ , Figure 11 has 10 links from every event state. Equation (13) is modified accordingly for this by replacing the 2 in the denominator of the left-side by the degree of belief quantisation.

$$\text{For point-valued fuzzy numbers: } \left(\frac{1}{10}\right)^N = \frac{1}{|A|} \quad \text{and} \quad N = \log_{10}|A| \quad (18)$$

$$\text{For interval-valued fuzzy numbers: } \left(\frac{1}{10}\right)^N = \frac{2}{|A|} \quad \text{and} \quad N = \log_{10}|A| - \log_{10} 2 \quad (19)$$

Thus, for fuzzy evidence as partial belief possibilities of crisp state values, the degree of belief quantisation determines the log base.

Equation (13) thus shows how the Hartley measure is dependent on crisp events or states. Although it may be fuzzified to allow for partial belief values, and extended to interval-valued variables as above, the concept rests on the choice of 1 or 2 values from a finite set of possible values. Next, the measure will be modified for events or states with *fuzzy* elemental values.

*Vague Interval Variables (PN)*

*(Fuzzy evidence, Fuzzy states)*

This class represents possibilistic numbers (PN) where a band of values as vague intervals are the elements of the set. The difference for this category is that the number of alternatives to choose from refers to the *number of possible intervals* at any alpha-cut, which is the aggregate number of intervals at, and above, that alpha value. This is the definition of Interval Cardinality (I).

$$\text{Then } \left(\frac{1}{10}\right)^N = \frac{1}{|I|} \quad \text{where } |I| \text{ is the Interval Cardinality} \quad (20)$$

$$\text{and } N = \log_{10}|I| \quad (21)$$

After presenting a general Specificity formula for TPD, the above concepts will be applied to develop modified and extended U-Uncertainty formulas for FN and PN triangular possibility distributions (TPD).

## 5.4 A General Specificity Formula for Triangular Possibility Distributions

A general formula for Yager's Specificity measure for fuzzy ambiguity will be first developed for triangular possibility distributions (TPD). The purpose of this is to demonstrate the aggregation of cardinality values within a fuzzy set.

Consider a FN Type TPD:

For any discrete or continuous set of elements there must be at least one element for the set to exist. Then the Cardinality  $|A|$  at any level of presumption ( $\alpha$ ) is given by:

$|A|_{\alpha} = 1 + \text{Interval}$ , where Interval is the  $\alpha$ -cut interval.

Following Yager's [109] definition -

$$\begin{aligned} \text{Specificity (S)} &= \int_0^1 \frac{1}{\text{Card}_{\alpha}} d\alpha \\ &= \int_0^1 \frac{1}{1 + \text{Interval}(\alpha)} d\alpha \end{aligned}$$

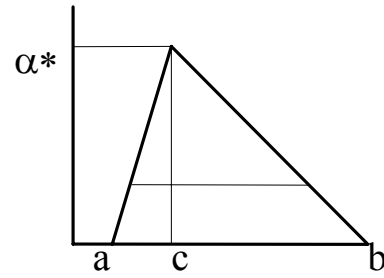


Figure 12: A FN TPD ( $a, c, b$ )

For a FN TPD, where  $\alpha^* : [0,1]$ , the interval at any alpha-cut is given by:

$$\text{Int}(\alpha) = (b - a) - \frac{\alpha}{\alpha^*} (b - a), \text{ which is independent of "c"}$$

$$\begin{aligned} \text{and } S &= \int_0^1 \frac{1}{1 + \text{Interval}(\alpha)} d\alpha \\ &= \int_0^1 \frac{1}{1 + (b - a) - \frac{\alpha}{\alpha^*} (b - a)} d\alpha \\ &= \int_0^{\alpha^*} \frac{1}{A - B\alpha} d\alpha \\ &\quad \text{where } A = b - a + 1 \text{ and } B = (b - a)/\alpha^* \\ &= -\frac{1}{B} \left[ \ln(A - B\alpha) \right]_0^{\alpha^*} \end{aligned} \quad (22)$$

Example: Let  $a=1, b=15, c=8, \alpha^* = 0.5$

$$\begin{aligned} S &= \int_0^{0.5} \frac{1}{1 + (14 - 28\alpha)} d\alpha \\ &= \int_0^{0.5} \frac{1}{15 - 28\alpha} d\alpha \\ &= -\frac{1}{28} \left[ \ln(15 - 28\alpha) \right]_0^{0.5} \\ &= -\frac{1}{28} (\ln 1 - \ln 15) = \frac{\ln 15}{28} = 0.097 \end{aligned}$$

## 5.5 Extension to Klir's U-Uncertainty Measure

Since the U-Uncertainty measure proposed by Higashi and Klir [46] is a direct extension of the Hartley information measure [43] which assumes a binary event space, the following modifications will be made to allow for non-binary event spaces as are required for both FN and PN fuzzy distributions. U-Uncertainty is defined for a TPD using log base 10 since belief levels of fuzzy sets will be treated as decimal quantised (as in Section 5.3). For FN, the difference between using log base 10 or natural logs, as in Klir and Yuan [58], is the presence of a constant factor ( $\log_{10} e = 0.434$ ) in (24). Thus the U-Uncertainty value cannot simply be converted between log bases by a constant multiplier, as is commonly stated, because 0.434 appears in a subtraction term in equation (24). But for PN variables with 2 dimensions of possibility, the modifications result in an even greater divergence from Klir's formula. The definition of a FN TPD(a,c,b) is as shown in Fig. 12, and to avoid confusion with a previous uncertainty abbreviation (U), a different abbreviation for U-Uncertainty will be used: Non-Specificity' (NSP'). (But neither should this be confused with the complement of Yager's Specificity measure NSP.) Klir [58] has also proposed a separate expression for U-Uncertainty based on the difference of successive possibilities, but this yields identical results to the cardinality formulation below.

*The Non-Specificity Measure for a FN Type TPD:*

$$\begin{aligned}
 NSP' &= \int_0^{\alpha^*} \log (Card_{\alpha}) \, d\alpha \\
 &= \int_0^{\alpha^*} \log (1 + Int(\alpha)) \, d\alpha \\
 &= \int_0^{\alpha^*} \log \left( 1 + (b-a) - \frac{\alpha}{\alpha^*} (b-a) \right) \, d\alpha \\
 &= \int_0^{\alpha^*} \log (A - B\alpha) \, d\alpha
 \end{aligned} \tag{23}$$

where  $A = b - a + 1$  and  $B = (b-a)/\alpha^*$

Integrating by Parts;  $\int u \cdot dv = u \cdot v - \int v \cdot du$

Put  $u = \log (A - B\alpha)$  and  $dv = d\alpha$  then,

$$\begin{aligned}
 \frac{du}{d\alpha} &= -\frac{B}{A - B\alpha} (\log_{10} e) \\
 du &= -\frac{B}{A - B\alpha} (\log_{10} e) d\alpha \\
 \text{and } v &= \alpha + C1
 \end{aligned}$$

$$\begin{aligned}
\text{Thus, } NSP' &= \left[ \log (A - B\alpha)(\alpha + C1) \right]_0^{\alpha^*} - \int_0^{\alpha^*} (\alpha + C1) \left( \frac{-B}{A - B\alpha} \right) (\log_{10} e) d\alpha \\
&= \left[ \log (A - B\alpha)(\alpha + C1) \right]_0^{\alpha^*} - 0.434 \int_0^{\alpha^*} \left( \frac{-C1B - B\alpha}{A - B\alpha} \right) d\alpha \\
\text{Let } C1 &= -\frac{A}{B} \\
NSP' &= \left[ \log (A - B\alpha) \left( \alpha - \frac{A}{B} \right) - 0.434\alpha \right]_0^{\alpha^*} \\
&= \left[ -\frac{1}{B} (A - B\alpha) \log (A - B\alpha) - 0.434\alpha \right]_0^{\alpha^*} \tag{24}
\end{aligned}$$

Example:  $a = 0, b = 4, (c = 2), \alpha^* = 1.0$

$$NSP' = \left[ -\frac{1}{4} (5 - 4\alpha) \log (5 - 4\alpha) - 0.434\alpha \right]_0^1 = \frac{5}{4} \log 5 - 0.434 = 0.439$$

*The Non-Specificity for a PN Type TPD:*

As outlined in Section 5.3, the cardinality of a PN at any belief level is taken to be the number of possible interval elements at that level and above. Thus, cardinality evaluation is different in a PN to that of a FN. Consequently, the Hartley measure is affected because the right side in equation (13), representing the identification of one value from a set of possible values, is affected by this different cardinality for the PN. For a PN the previously described first approximation for the number of interval elements is the number of values on the left side of the modal value (c). So the number of possible interval elements *at any*  $\alpha$  is the total number of feasible values  $\geq \alpha$ , which is the *upper portion* of the area on the left side of (c) at that  $\alpha$  level (Figure 13). For non-symmetrical TPD, the exact NSP measure requires integration over both sides of the mode and an average taken. The deviations between the exact and approximate versions of NSP will now be demonstrated.

Then for a PN the approximate Cardinality  $|I|$  at any  $\alpha$  is:

$$\begin{aligned}
|I|_{\alpha} &= \left( \frac{\alpha^* - \alpha}{\alpha^*} \right) \left( \frac{1}{2} (c - a) \alpha^* \right) \\
&= R - S \alpha
\end{aligned}$$

where  $R = (c - a) \alpha^* / 2$ ,  
and  $S = (c - a) / 2$

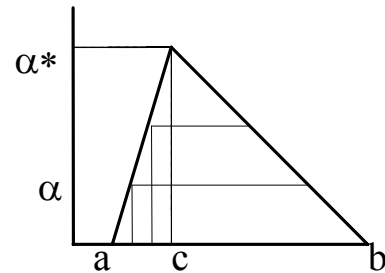


Figure 13: A PN TPD( $a, c, b$ )

Then the approximate NSP/ for a PN :

$$\begin{aligned}
 \text{NSP/} &= \int_0^{\alpha^*} \log (\text{Card}_\alpha) d\alpha \\
 &= \int_0^{\alpha^*} \log (R - S\alpha) d\alpha \\
 &= \left[ -\frac{1}{S}(R - S\alpha) \log (R - S\alpha) - 0.434\alpha \right]_0^{\alpha^*} \quad \text{as per (24)}
 \end{aligned}$$

And the exact NSP/ for PN as the average of left and right side integrals is:

$$\text{NSP/} = 0.5 \left[ \int_0^{\alpha^*} \log (R - S\alpha) d\alpha + \int_0^{\alpha^*} \log (P - Q\alpha) d\alpha \right] \quad (25)$$

$$\text{where } P = (b-c) \alpha^* / 2 \quad \text{and } Q = (b-c) / 2$$

For a test sample of TPD expressions (1-8) below, the approximate NSP/ values are shown below, with results for exact and approximate versions compared in Table 8.

Example Non-Specificity Calculations for PN using Left-side Approximation :

1. a=0, c=2, b=4, $\alpha^* = 1.0$	NSP/ =	$\log 1 - 0.434$	= -0.43
2. a=0, c=3, b=4, $\alpha^* = 1.0$	NSP/ =	$\log 1.5 - 0.434$	= -0.25
3. a=0, c=3, b=4, $\alpha^* = 0.5$	NSP/ =	$.5 \log 0.75 - 0.217$	= -0.28
4. a=0, c=8, b=20, $\alpha^* = 1.0$	NSP/ =	$\log 4 - 0.434$	= 0.17
5. a=0, c=10, b=20, $\alpha^* = 1.0$	NSP/ =	$\log 5 - 0.434$	= 0.26
6. a=0, c=50, b=200, $\alpha^* = 1.0$	NSP/ =	$\log 25 - 0.434$	= 0.96
7. a=0, c=100, b=1000, $\alpha^* = 1.0$	NSP/ =	$\log 50 - 0.434$	= 1.26
8. a=25, c=75, b=1000, $\alpha^* = 0.7$	NSP/ =	$.7 \log (25)(.7) - 0.304$	= 0.57

**Table 8:** Extended U-Uncertainty Measure Results

Example Number	1	2	3	4	5	6	7	8
FN	0.44	0.44	0.22	0.95	0.95	1.88	2.57	1.79
PN (Approx. NSP/)	-0.43	-0.25	-0.28	0.17	0.26	0.96	1.26	0.57
PN (Exact NSP/)	-0.43	-0.49	-0.40	0.25	0.26	1.20	1.74	0.95

It can be noted that U-Uncertainty, as ambiguity information, is always less in PN than FN, as is to be expected since it is the minor form of uncertainty present in PN. Also, the FN values are independent of the modal position (c), whereas the PN values are not independent and are a function of the skewness of the fuzzy sets. Thus, when the U-Uncertainty formula has been modified for the Cardinality of PN interval elements, the measure itself has less relevance since ambiguity is the minor form of U present. But even for FN, which equate to traditional fuzzy sets, the derived formula with log base 10 produces different results to the U-Uncertainty expression with natural logarithms as frequently appears in the literature. And as noted previously, this choice of log base is not arbitrary, and a multiplier cannot simply convert between two different log base results as if there were is no constant in expression (24).

## 5.6 Section Summary

This section has examined the foundations of some well-known fuzzy U measures. A new approach to applying the probabilistic entropy concept to the fuzzy domain was initially presented based on capturing information on variations, of both real state values and value belief grades. This addressed a fundamental weakness inherent in many transpositions of the entropy concept to the fuzzy domain because they implicitly assume binary state variables. Next, the Hartley Information concept was extended to the fuzzy domain for both FN and PN type variables, and the information entropy construct was developed by induction highlighting the meaning of the log base as the degree of quantisation of belief. In light of the fundamental development of the Hartley measure, it was shown how the information entropy construct is meaningful only for variables with binary belief states. Thus it must be very carefully adapted when applied to the domain of fuzzy information where partial belief levels of variable state values are present. General formulae for computing U-Uncertainty for TPD were also developed for both FN and PN. Using a range of example TPD, it was shown that the U-Uncertainty measure is less informative for PN type vague variables, because it is fundamentally an ambiguity measure. Furthermore, the prevalent U-Uncertainty form in the literature yields divergent results even for FN because it applies natural logs which cannot be simply converted to the more correct log base 10. A general formula for Yager's Specificity for TPD was also developed to demonstrate the different type of cardinality evaluation required for FN with point-valued elements, compared to PN with interval elements. These extensions have been presented to show how some of the measures of fuzzy U that are commonly found in the literature can only address the ambiguity form of fuzzy U since they are based on the set *cardinality* concept. Furthermore, some are also susceptible to errors caused by transposing crisp probabilistic concepts to the fuzzy domain.



## 6. Estimation with Hybrid Uncertainty

### 6.1 Introduction

Condensing sets of soft information with hybrid U forms into value estimates for variables in a model presents a number of challenges. The objective of Section 3 was to identify the precise nature of the variable in which a mixture of U sources are present so that the full U content could be managed in system computations. Subsequently, Section 4 was concerned with suitable and adequate methods to quantify the forms of fuzzy U present in a fuzzy variable after its possibility distribution has been given or developed. This section now investigates methods to condense different sets of soft information into value estimates, whether a crisp singleton, crisp interval, or fuzzy set. A wide variety of methods can be found in the literature, and frequently they appear to be competitive leading to conflicts of opinion. To commence untangling this rather confusing subject, a range of methods that appear in the literature will be classified according to the category of hybrid U they best suit. The objective is to put the available methods into the context of the exact nature of the information available, and while this may sound rather fundamental, it is seldom done when these methods appear in the literature. Following that, a new type of fuzzy representation will be introduced that can also embed a limited amount of statistical information: the Stochastic Fuzzy Quantity. This type of variable is aimed at situations where some data about a rare or unexpected event exists but is of a sparse nature. Finally, a new approach to representing the total U in a variable as a crisp interval is introduced. This method allows both statistical and non-statistical forms of U to be condensed into a uniform representation which can then be handled in computations by the standard interval algebra of Moore [70].

### 6.2 Measures of Typicality for Soft Information

In accordance with the taxonomy of uncertainty used throughout this work, three major hybrid categories of uncertainty are identified based on the degree of statistical information available, and the type of fuzzy variable induced by the uncertainties present in the information. We will now discuss briefly how the analysis of hybrid uncertainty forms can be of assistance when determining a measure of typicality, such as a fuzzy expected value (FEV) or a fuzzy estimate from a body of soft evidence. In a complex decision system such as a fuzzy expert system, it may be necessary to reduce the information granularity of a body of evidence to a measure of typicality which can then be represented in a standard form such as a trapezoidal fuzzy number.

But first the difference between an expected value and a typical value should be reviewed. A “typical” value is a representative value as a general tendency among a set of values. An “expected value” by definition is a function of values and their relative frequencies. It is not necessarily a representative value as it may be greatly affected by a few outlier values. So even for statistically sufficient sets of crisp data, some of the methods below may be applied to determine a typical value when outlier effects are not desirable. Friedman et al [37] have compared the behaviour of the Fuzzy Expected Value [36], the Clustering Fuzzy Expected Value [97], and their recent Most Typical Value method [37] for a range of crisp data sets. They conclude that the Most Typical Value method is more reasonable since it avoids some of

the irregularities of the other methods. The following methods for determining a FEV have been proposed over the last 20 years or so, and are selected because of their relationship to the hybrid forms of fuzziness that can be found in real-world data. However, no evaluation of their respective theoretical merits is implied.

**Table 9:** *Types of Fuzziness in Evidence*

		EVIDENTIAL UNCERTAINTY	
		STATISTICALLY SUFFICIENT	STATISTICALLY INSUFFICIENT
VARIABLE TYPE	CRISP	P	TYPE 2 $\tilde{P}$
	FN	TYPE 1 Pr (V is FN) is P	TYPE 3 Pr (V is FN) is $\tilde{P}$
	PN	TYPE 1 Pr (V is PN) is P	TYPE 3 Pr (V is PN) is $\tilde{P}$

The most common example of a FEV computation is reducing a *single fuzzy set* to a single crisp value as may be used to modify a control variable. In this case there is no multiplicity of evidence or random variations and a variety of defuzzification methods are available. The most common method is the Centroid or Centre of Gravity method, although the Centre of Area method would seem to be more appropriate to the ordinal information in a possibility distribution. A wide range of literature exists concerning defuzzification methods but they will not be discussed in this report. While a singleton output may be appropriate in control applications, it may not be appropriate if the variable is not so restricted, as is the case for the utility value of an alternative in multicriteria decision making. In this case, perhaps computing an expected *fuzzy interval* would be more suitable. In this regard, Gonzalez and Vila [41] and Heilpern [45] have proposed methods for determining such a *real interval* from a single fuzzy set. However, the following discussion will identify more rigorous algorithmic methods for determining a measure of typicality from a careful analysis of the hybrid U forms present in a body of soft information. Table 9 summarises the classes of hybrid uncertainties used throughout this report and identifies three main hybrid U categories: Types 1, 2, and 3. A selection of methods appearing in the literature for determining measures of typicality for soft information will now be associated with these three categories of hybrid information.

### 6.2.1 Soft Information Type 1

#### *Random Fuzzy Variables*

Fuzzy Variable (FN or PN) / Statistically sufficient sample       $\Pr(x \text{ is } \tilde{A}) \text{ is } P$

Hybrid uncertainty of Type 1 is evidence with statistically sufficient information on the variability of a fuzzy variable ( $\tilde{A}$ ). The mean truth-value of the fuzzy identity ( $x \text{ is } \tilde{A}$ ) can be estimated from the frequency distribution of fuzzy values with the fuzziness of elements represented in one of two forms.

The first form is when the fuzziness in elements is represented as a standardised fuzzy distribution such as a triangular or trapezoidal fuzzy distribution. In this case, the characteristic function of the FEV will be derived by extended (fuzzified) algebraic operations to aggregate the fuzzy data. If the estimates are all TPD for example, the FEV will be the algebraic average TPD. But if the elements within the sample require weighting with fuzzy weights, an algorithm for non-linear optimisation such as the Dong and Wong algorithm [22] is then appropriate. Both PN and FN can be averaged in this manner since the FEV is dependent only on the non-fuzzy variability information in the evidence. Another type of fuzzy information in the evidence is when the values of the elements are numbers with fuzzy constraints, but without a standardised possibility representation. Examples of this kind of ambiguous evidence [82] are:

5 people earn *between* \$10 - \$15  
 7 people are *about* 30 years of age  
 9 people are *not more* than 160 cm height

With data sets of this type of fuzzy ambiguity, a characteristic function must be used to initially fuzzify the information before the FEV can be derived algorithmically. Methods such as this for developing the FEV from given *a priori* characteristic functions, are Schneider and Kandel's FEV [81] and Friedman et al Weighted FEV [36]. More recent clustering based methods are the Vassiliadis et al [97] Clustering FEV and the Most Typical Value of Friedman et al [37].

### 6.2.2 Soft Information Type 2

#### *Fuzzy Random Variables*

Crisp variable / Statistically insufficient sample       $\Pr(x \text{ is } A) \text{ is } \tilde{P}$

Hybrid uncertainty of Type 2 (which does not mean the traditional Type 2 fuzzy set) is evidence where the elements are crisp, but there is sparse statistical information resulting in estimates that are necessarily fuzzy values. This category relates to fuzzy probabilities of which there are a variety of definitions. Zimmermann [119] discusses a selection of these definitions including the earliest form by Zadeh [116] to be discussed below

A sample may be statistically insufficient in two ways.

#### Small Number of Elements:

In this case the evidence yields fuzzy probabilities (FN) and it would seem that Zadeh's early definition [116] is appropriate,  $p(x) = \int \mu_{p(x)} p(x) dp(x)$ , whereby for any value  $x$  there are a range of  $p(x)$  and the integral determines the net probability overall for each  $x$ . However, the problem with this is that a characteristic function for  $p(x)$  cannot be determined from a limited amount of evidence, and even when there is sufficient statistical evidence, it is no trivial problem to determine a characteristic function from relative frequencies even though some methods are available [48]. An expedient solution may be to fit a standard membership function to the estimate using some guesstimates about the modal position and degree of incompleteness of the data. A defuzzification method could then be used to reduce this fuzzy set to a *singleton*, which is what the FEV for such a crisp variable should be.

#### Non-Representative Elements:

In this case, there is a higher-level of  $U$  because the sample does not sufficiently cover (i.e. represent) the universe of discourse and the evidence induces a PN possibilistic likelihood ( $\tilde{P}$ ) rather than a FN probabilistic likelihood. Similar methods to above for sparse data must also be used to develop a characteristic function, but in this case, the defuzzified measure of typicality should be an *interval* value rather than a singleton. The aforementioned methods of Gonzalez and Vila [41] and Heilpern [45] are again suitable for determining such a typical interval from a characteristic function.

### 6.2.3 Soft Information Type 3

#### *Fuzzy Variable / Fuzzy Sample*

Fuzzy Variable (FN or PN) / Statistically insufficient sample

$\Pr(x \text{ is } \tilde{A}) \text{ is } \tilde{P}$

Hybrid uncertainty Type 3 represents statistical insufficient information on a fuzzy variable (as may be induced by approximate data). It thus embodies more dimensions of pre-computational uncertainty and suitable methods for determining a FEV depend on the amount of information available about the characteristic functions of both  $\tilde{A}$  and  $\tilde{P}$ . The Fuzzy Expected Interval (FEI) of Schneider and Kandel [81] is one approach when there is only approximate information on the fuzziness. With this method linguistic modifiers are used to quantify the approximate counts (which cause the ambiguous probabilities) and these are combined with a characteristic function based on some limiting values to derive the typical interval estimate (FEI). The following example of such evidence is taken from [82]:

More or less 30 people	earn \$2.50
50 people	earn between \$4 and \$5
70 to 100 people	earn \$5.50
50 to 70 people	earn \$7 to \$8

What is the expected hourly rate (FEI) of a person?

With Input Membership Functions ( $\tilde{A}$  and  $\tilde{P}$ ):

When more information on the fuzziness is available in the form of the membership functions for  $\tilde{A}$  and  $\tilde{P}$ , the characteristic function (or a non-standard membership function) of the typical value may be derived. Chanas and Florkiewicz [14] develop one such algorithm. Another method for this level of information has also been developed by Yager [110].

In conclusion, this section has associated several methods that appear in the literature for computing a measure of typicality from soft evidence with three types of hybrid uncertainty. The identification of the uncertainty hybrid, in conjunction with the amount of information available on the forms of fuzziness, then assists with the selection of a suitable technique for the determination of a measure of typicality for the body of soft evidence.

### 6.3 A Stochastic Fuzzy Representation

The usefulness of a new type of fuzzy representation with some stochastic information also embedded will now be demonstrated by means of an example. This fuzzy representation will be called a Stochastic Fuzzy Quantity (SFQ). The type of variable this construct applies to are quantitative variables that are numerically evaluated but for which there is only sparse or sporadic data perhaps pertaining to unwanted, unusual, or exception states. Such data sets may consist of event counts, state values, or time durations.

Consider an important communications network for which a performance index (PI) is updated each month based on the number of hours the network has been dysfunctional each day during the month. The objective is to derive one index  $[0,1]$  summarising the performance over the month based on deviant behaviour. A difficulty arises with the fuzzification of the time-duration data since the set of values actually represents *exception* information and each month there are normally a *large proportion of zero* or small values plus a small number of more significant values. Thus, these zero or small values should be discounted in the monthly performance estimate so that they do not diffuse the information on the exception states. If standard statistical estimation procedures were to be used an expected value could be derived by weighting values with frequencies so that many small values would have the same contribution to the estimate as a few large values. Furthermore, the standard deviation statistic estimates an average squared dispersion about the sample mean, but the *changing number* of significant elements in the sample sets attenuates the exception information provided by such an average deviation estimate. Even if the sum of squared errors itself (i.e. *not* averaged to eliminate the effect of a variable number of elements) were used to this end, the non-linear sensitivity to deviation values would be too extreme. These considerations, plus the fact that no statistical distribution (such as Student-t) can be assumed for this data in order to estimate the likelihood of a value range severely limits the application of statistical methods.

As a practical approach for summarising the monthly data to highlight the *time in the exception states*, the SFQ is proposed which embeds some stochastic information into the possibility distribution. It is the qualitative nature of the possibility measure itself that allows some stochastic information to be introduced in this way. This approach avoids the loss all frequency information within the month which results when a TPD is simply formed from a data set by setting the range of the data values as the TPD support or base of the triangle. This

SFQ method is based on two data features: the spread of values about the median, and the *proportion of non-zero values*. Although, there are many notions in the literature for transforming between probability and possibility distributions, simply speaking, some consistency requirements must be satisfied, which vary from weak or minimal requirements, to general requirements based on uncertainty management principles [57]. The minimal requirement is  $\text{Pos}(x) \geq \text{Prob}(x)$  (the Consistency Principle [117]) and this is the only requirement satisfied by the following method. It is used for determining the right-side of the TPD from the probability of a *conditional average*. For this reason, the right-side of the SFQ represents the lower boundary of an unknown possibility distribution, and consequently, the COA of the SFQ is the *lower* limit for the unknown possibility distribution.

The desired monthly performance index (PI) is based on an allowable range of COA values. Maximum performance (PI=100) is attributed to no exception state occurrences, or COA = 0. Minimum performance (PI=0) is an input limit (Z) beyond which performance is considered as zero. The monthly PI is then based on the COA of the updated SFQ:

$$\text{PI} = \frac{Z - \text{COA}}{Z} \quad (26)$$

Since the COA is based on the lower boundary of an unknown possibility distribution, the PI will be an upper performance limit. But such an optimistic estimate is reasonable because the stochastic information more accurately positions the boundary on the right-side slope of the TPD. This combination of different types of information, while not violating any basic principles, enables a performance measure to be derived which provides more information than if the simple range were used for the support of the TPD. The SFQ computational procedure is as follows.

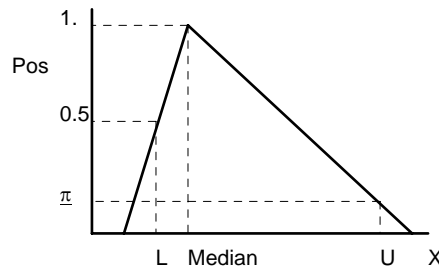
#### *Developing the Stochastic Fuzzy Quantity TPD*

1. Order the monthly set of daily data
2. Determine the Median or middle value, or average middle two if there is an even number. This is used for the mode or apex of the SFQ (b).
3. Average all non-zero values  $\Rightarrow U$
4. Average all values below Median  $\Rightarrow L$  (including zeros)
5. Determine relative frequency of non-zero values ( $\underline{\pi}$ )  
 $= \text{Number non-zero values} / \text{Total number}$
6. Let  $\text{Pos}(L) = 0.5$  (because determined from lower half of total number)

The value and possibility pairs (L,0.5; Median,1.0; U, $\underline{\pi}$ ) then define the quantitative SFQ as in Figure 14, and the PI for the month is derived from the defuzzified Centre of Area (COA) of the SFQ. The COA thus represents a summary of the composite statistical and fuzzy information for the month. This is considered to be preferable to the Centroid defuzzification method since it is purely based on ordinal belief considerations, which is what possibility distributions fundamentally display. As previously noted the centroid expression has no meaning for fuzzy distributions, as compared to probability distributions where moments do have meanings.

However, the three points (U, Median, L) do not necessarily represent a simple triangular distribution and the forms that may be defined are:

- a complete triangular possibility distribution (TPD) when  $L < \text{Median} < U$
- a right-angled triangle when  $\text{Median} = L$  or  $U$
- a left truncated TPD when left side intersects zero value axis  
(the TPD is never truncated at right side)
- a right ( and possibly left) truncated trapezoidal possibility distribution  
when  $U < \text{Median}$  ( $L < \text{Median}$  always by definition )



**Figure 14:** The Stochastic Fuzzy Quantity

A general formula is developed in Appendix A for calculating the COA for SFQ with left and right truncated TPD, although only left TPD truncation can actually appear in the SFQ. (When truncated *trapezoidal* SFQ are encountered instead of TPD, the COA can be determined by using this general formula for the left triangular area under the trapezoid and displacing it to the right by half the base of the right-side rectangle.) For comparative purposes, in the following examples the simple arithmetic average of the data, and its corresponding PI are shown in brackets below the SFQ derived values for PI. Example 2 also demonstrates a negative SFQ performance value in brackets where the COA falls below the limit for zero performance.

Example Quantitative Data Sets : ( 30 day sets, Units decimal hours say)

1.  $\{24 \times 0, 5 \times 6, 1 \times 25\}$  Minimum Performance Limit(Z) = 10  
 $\Rightarrow \text{Median} = 0, U = 9, L = 0, \underline{\pi} = 0.2$   
SFQ: COA = 3.8, **PI = 0.62**  
(Avg. = 1.8, PI = 0.82 )
2.  $\{27 \times 0, 3 \times 50\}$  Minimum Performance Limit(Z) = 15  
 $\Rightarrow \text{Median} = 0, U = 50, L = 0, \underline{\pi} = 0.1$   
SFQ: COA = 19.0, **PI = 0** ( - 0.26 )  
(Avg. = 5, PI = 0.66 )
3.  $\{15 \times 0, 15 \times 10\}$  Minimum Performance Limit(Z) = 15  
 $\Rightarrow \text{Median} = 5, U = 10, L = 0, \underline{\pi} = 0.5$   
SFQ: COA = 5.6, **PI = 0.62**  
(Avg. = 5, PI = 0.66 )

These examples demonstrate how a performance measure PI can be determined using exception information in a manner which sensitises PI non-linearly to non-zero values across the range of values (and which is also biased towards higher values).

## 6.4 The Composite Uncertainty Interval

In some types of models, such as complex system reliability analysis (to be covered in Section 7) the FN and PN types of fuzzy variable can be separately managed with operators appropriate to the U forms they contain. However, in other types of models this may not be convenient, nor indeed possible, and in these cases another approach may be adopted. This alternative approach will condense the different forms of uncertainty in a fuzzy quantity into a crisp interval representing the range of real values possible. Such an interval will be termed here a Composite Uncertainty Interval (CUI) and can be derived using the new Ambiguity and Vagueness measures that have been proposed. In contrast to Kaufmann and Gupta's [53] hybrid number concept, which consists of a fuzzy number plus a *separate* statistical component, the composite interval *integrates* the U forms present into a real number interval. The previously defined Ambiguity and Vagueness measures can provide information on the proportion of each type of fuzzy uncertainty present in a fuzzy set, in relation to the COA of the set (as distinct to the Measure of Fuzziness which relates more to the shape characteristic of a fuzzy set). The sum of the Ambiguity and Vagueness percentages then represents the total (T%) fuzzy uncertainty *information* embedded in the possibility distribution. Assuming a symmetrical U spread about the COA, the condensed real interval is simply the COA of the distribution  $\pm T/2$  %. Algebraic and logical operations can then proceed using Moore's [70] interval arithmetic, and in this manner different U forms are reduced to a unified representation.

The procedure can be illustrated using a simple example relating to military communication system design. The objective is to compare structural alternatives on the basis of their reliability, vulnerability, and cost. While increasing the number of network nodes can increase the system reliability, it can also increase the vulnerability of the system. And since some subjective estimates will almost certainly be required for the occurrence of rare, or never-have-occurred failure events, higher-order uncertainty PN estimates will be induced. Furthermore, for each system design alternative there is also a lifetime cost (maybe not so critical for combat systems but more important for operations support systems). This is also a vague or possibilistic variable due to the uncertainty in lifetime duration estimates, interest rate variations, and other assumptions. Let all variables be normalised [0,1]. Example CUI for three alternative system designs are shown in Table 10 after normalising and aligning scales. It can be seen that no alternative is totally dominated by any other and this problem is the standard multi-objective trade-off rationalisation problem. To address this type of tradeoff problem, a new method called "Globular Knowledge Fusion" (GKF) has been proposed by this author [102,103]. By that method a global value is derived for each alternative according to the desired sensitivity to the disparities between the three different performance metrics, and the "best" alternative is then based on this global or integrated metric. Since GKF only requires



addition and multiplication operations, it can also be used with the application of interval algebra.

**Table 10:** *Composite Uncertainty Intervals for an Alternative Selection Problem*

	Reliability	Vulnerability	Cost
Alternative 1	[0.90, 0.93]	[0.70, 0.75]	[0.20, 0.29]
Alternative 2	[0.75, 0.83]	[0.77, 0.88]	[0.43, 0.47]
Alternative 3	[0.69, 0.74]	[0.84, 0.90]	[0.61, 0.72]

## 6.5 Total Uncertainty in a Variable

The quantification of total uncertainty is naturally dependent on the conceptual framework used to represent and understand hybrid uncertainty combinations. This section will present an approach that is based upon the descriptive framework introduced in Section 3. But before that method is introduced, a brief summary will be presented of other approaches in the literature to breaking down the big-picture of hybrid U combinations. Klir [56] has offered the term “General Information Theory” to describe the mix of recent methods developed to model higher-order U forms that classical probability theory cannot adequately model. These methods include Dempster-Shafer theory of evidence, random set theory, possibility and fuzzy measure theory, rough sets and gray theory. Klir [54] and Dubois and Prade [24,27] have also presented relational structures that attempt to place some of the above methods into the context of different U forms. Joslyn [47,49,50] has also investigated taxonomic structures based on random set theory. However, none of the above authors have developed measures which could be considered to evaluate all the potential forms of U that comprise total U in a variable. Most commonly, total uncertainty (T) is evaluated as:

$$T = \text{Statistical uncertainty} + \text{Fuzzy uncertainty}$$

$$\text{where Fuzzy uncertainty} = \text{Non-Specificity} + \text{Fuzziness}$$

Some authors attempt to sum measures for the different U forms above with different units. As previously stated, Pal and Bezdek [76] have pointed-out the potential danger of combining measures of different forms of U with different units. Another group of authors [4,61] attempt to develop fuzzy entropy measures (or a hybrid entropy measure [39]) for a fuzzy variable and then combine it with the Shannon probabilistic entropy (i.e. with the same units). But over and above the problem of measurement units, there is a deeper problem associated with applying the probabilistic entropy function to a fuzzy set. This problem is that the substitution of the fuzzy metric ( $\mu$  or  $\pi$ ) into the entropy expression is meaningless because, by definition, the function depends on the U metric being an aleatory probability or relative frequency. Thus all simple transposes of the entropy measure to fuzzy sets as previously described are meaningless. Other authors (such as Klir [54]) sometimes add a measure for the conflict in the evidence to the total fuzzy U quantification. However, no separate measure for conflict is required in the following procedure since differences within evidence are captured and quantified by the Ambiguity measure.

One of the difficulties in trying to determine a total uncertainty measure is that there can be several types of stochastic U in addition to the different types of fuzzy U within a set of information. Take for example, information supplied from a surveillance operation on an adversary's forces. Stochastic U may be introduced due to the randomness in the movements of the adversaries forces ( $r_1$ ), or in the observation process itself ( $r_2$ ), or in the interpretation of the observation process as cognitive variations within a single observer ( $r_3$ ).

Consider the following cases:

**Case 1:** Observation window is small,  
(20 minutes with seven separate sensor observations)

In this case,  $r_1$  and  $r_3 = 0$ ,  $r_2 \neq 0$ . This sparse data set can be used to derive a Stochastic Fuzzy Quantity (TPD) with some small degree of statistical information embedded. This TPD would then be interpreted as a FN type fuzzy set with only the ambiguity due to fuzziness caused by observation process imprecision.

**Case 2:** Input set is 1 month of daily sensor observations of a single variable.

Here  $r_1$  and  $r_2 \neq 0$ ,  $r_3 = 0$ . This data set presents more statistical information and a statistical U measure can be combined with a fuzzy U measure. After fixing a standardised level of confidence (say 95%) a crisp interval for  $r_1$  can be estimated using conventional statistical methods. In addition, a FN type fuzzy set can be derived (say a standard TPD) from the same data set representing the observation imprecision ( $r_2$ ). There are a variety of methods for deriving a membership function from a data set, according to the type and amount of information available. Although many of these derive non-standardised types of membership functions, it may still be possible to fit some standardised form to them. For example, Joslyn [48] has presented a relatively straightforward method based on histograms. However, the Stochastic Fuzzy Quantity of this report may also be an appropriate method to condense sensor (SIGINT) information into a TPD when the data set contains many negligible observations and only a few significant detections, as opposed to a complete set of quantified evaluations. Recall that the SFQ has been developed to sensitise a measure to the occurrence of rare or exception events. As well as applications in modelling defect data for process quality control, a similar type of sensitivity may also be useful in surveillance operations where SIGINT streams contain relatively few positive observations. After deriving such a TPD, the Ambiguity measure previously described may then quantify the proportional U due to ambiguity and a crisp possibility CUI formed. Since both statistical and fuzzy U intervals are derived from the same data set, their combined interval should be formed by taking the Min and Max over both intervals, whether they overlap or not (any separation between respective intervals being caused by outliers).

**Case 3:** A human agent (HUMINT) has provided a set of 30 observations over one month.

In this case, human cognitive U may be also present ( $r_1, r_2, r_3 \neq 0$ ) and the TPD derived above as a SFQ is a vague PN fuzzy quantity. Then vagueness and associated ambiguity must *both* be quantified and the TPD reduced to a crisp CUI as previously described. This can then be

combined with the stochastic U interval to form a total U interval estimate based on the three different types of uncertainty (as in Case 1).

It should be noted that the use of any form of entropy measure to evaluate stochastic U is avoided by this approach. The primary reason for this is to ensure compatibility of units so that a meaningful aggregate can be developed. However, the new fuzzy entropy measure that was presented in Section 5.2 may be useful in some partitioning or clustering type applications. Indeed, several fuzzy entropy measures of questionable derivation have already found some utility in such applications (i.e. better than nothing). So the CUI has been presented above as a pragmatic approach for developing crisp estimates from hybrid U data sets. As well as enabling hybrid U aggregation based on an explicit and easily understood rationale, it also has the advantage of enabling subsequent higher level computations through the mechanism of interval mathematics.

## 6.6 Section Summary

This section has introduced a number of techniques for condensing soft information sets with hybrid U forms into single estimates, as Measures of Typicality, for subsequent system computations. Several techniques existing in the literature were initially presented in relation to three categories or types of hybrid uncertainty. The Stochastic Fuzzy Quantity was then introduced for situations where a fuzzy representation is required for a sparse data set, and its operation demonstrated when a variable's definition concerns the occurrence of rare events. Finally, the Composite Uncertainty Interval was introduced for situations where it is not desirable to work with FN and PN fuzzy quantities separately in computations. In such cases, the Compound Uncertainty Interval as a crisp interval can also provide a mechanism by which statistical U can be combined with different fuzzy U forms and processed systematically through interval algebra. This method also eliminates the problem of U measure units which plagues many other attempts in the literature to develop a representation for total U in a variable. Overall, these various soft information reduction methods address the different U forms that may exist in sample evidence (Level 5 Uncertainty in the framework of this report).

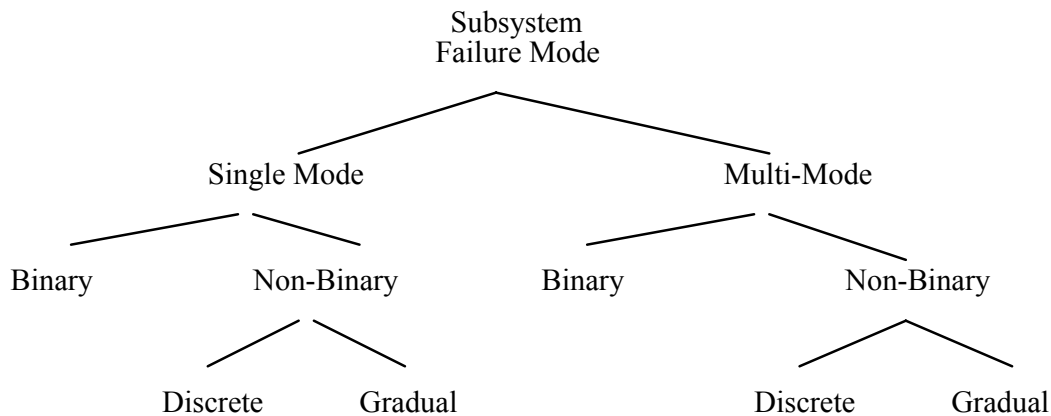
## 7. Reliability Analysis of Hybrid Uncertainty Systems

### 7.1 Introduction

The application of the proposed methods for hybrid U representation and management in complex models will now be demonstrated in the field of reliability analysis. As an operational performance metric for physical systems, reliability enables performance to be evaluated against design objectives and is of major importance in man-machine systems. Events as at the Chernobyl nuclear facility have indicated that the reliability of complex man-machine systems still leaves a lot to be desired and traditional methodologies for assessing and monitoring reliability do not appear to be achieving their goals. As a result, some new approaches have emerged which allow additional forms of soft information to be included, as may relate to human caused error. In this manner, several authors have applied fuzzy set theory (FST) to reliability analysis. However, the present author feels that many of those fuzzy investigations have practical limitations, and furthermore, certain theoretical results appear to violate fundamental uncertainty management principles. To address these aspects, a generalised, coherent, and practical approach to soft reliability analysis will be developed using the fuzzy quantity concepts of this report. This approach will be demonstrated for the major techniques of structure function analysis (the synthesis of physical system behaviour from the individual component lifetime characteristics) and fault tree analysis (reliability estimation from the potential occurrence of a variety of faults and their combinations which may be component failure or human error). To commence, a real-world system degradation characteristic will be discussed that has presented difficulties to traditional reliability modelling methods.

### 7.2 Modelling Non-Binary Failure Events

The failure, or degradation, event is the fundamental element of any reliability analysis and a key difficulty is how to model the type of multi-state or gradual failure events that are pervasive in real world systems. In fact, the widespread existence of such non-binary failure events is one reason that preventive and predictive maintenance is able to avoid system shutdowns due to unexpected failures, by being able to pre-empt failure after inspection of certain physical properties. Over the last decade or so there have been various attempts to model this aspect statistically, but as yet, restrictive assumptions and computational complexity have limited their application. Possibility theory has been applied by Cai [9,10,11,12] to this problem of fuzzy failure events, but those results are also rather limited (to be discussed later). Using the fuzzy quantity concepts previously described, an approach to modelling this kind of non-binary failure will now be presented that is based on identifying what types of multi-state failure events are present (Figure 15), and consequently, what types of fuzzy lifetime estimates are induced. Multi-state failure events exist when any failure mode of a component is non-binary. For this analysis a component refers to the smallest restoration unit (repair or replacement) of a system, as defined by the restoration practice for that system. Definitions of the key terms to be used now follow.



**Figure 15:** *Types of Failure Events*

#### Failure Mode:

A single component may have several different modes or methods of failure, each with its own characteristics. For example, an electric motor may fail by bearings, brushes or armature burn-out. Similarly, computer nodes may be degraded in multiple ways. If all modes are binary, the aggregate mode is also binary. But if any mode exhibits non-binary failure (discrete or gradual), the aggregate failure characteristic for the component is also non-binary, even though binary failure modes may also be present.

#### Failure State:

Any mode of failure may occur through a number of degradation degrees until performance reaches an unacceptable limit termed “failure”. Discrete degradation occurs through finite steps while gradual deterioration occurs through a continuous decrease in performance level. Binary failure  $\{0,1\}$  is when there are no steps in the degradation process.

A fuzzy lifetime estimate is induced by any non-binary failure data, and will have inherited  $U$  dimensions at least equal to those inherent in the data due to the difficulty or accuracy of failure identification at inspection. When accurate inspection or measurement of a physical covariate can reasonably predict failure because of known limits to the physical characteristic, the approximate event can be considered to induce a FN lifetime estimate with one dimension of possibility. This situation applies where failure can be approximately predicted from the physical condition with reasonable confidence. On the other hand, when inspection of the physical property can only be qualitatively or subjectively performed by the expert, a PN estimate will be induced with two dimensions of possibility due to this higher-order uncertainty. This may occur due to the nature of the characteristic, or from some physical constraint or obstruction to measuring, or from unknown limits on the physical property for failure. These constraints result in a lower confidence in failure prediction from the imperfect physical inspection. In this manner, the accuracy of identifying the degree of partial failure determines the type of fuzzy variable induced by the data on approximate failure times. The type of fuzzy variable then allows the effect of partial failures to be modelled very simply by using operations appropriate to the dimensions of possibility present in the respective types of fuzzy variable induced.

But superimposed on these determinants is the statistical sufficiency of the data set itself. As previously noted, if the data points can be considered to be a representative sample, but are sparse, a FN is induced. But if the set is not representative then a PN is induced. So even with accurate inspection and confident failure predictions, the data may still induce a PN. In this manner, non-binary (or fuzzy ) failure events may be modelled by identifying the type of fuzzy variable induced from the combination of component characteristics, inspection accuracy and amount of information available. However, methods for estimating the component lifetime characteristic function from fuzzy inspection data are beyond the scope of this report.

### 7.3 Structure Function Synthesis

This widely applied form of reliability estimation is used in the design of physical systems to ensure that a specified performance level is achieved, often through the medium of in-built redundancies. A fundamental assumption in Structure Function Synthesis (SFS) is that all systems are coherent systems, where each component has a purpose and a measurable contribution to system performance. If a component were not needed it would not be there, and for this reason, no component failure characteristic can be ignored in the evaluation of total system performance or reliability. In SFS, the component lifetime estimates are usually failure rates as the likelihood of failure event occurrence, or the inverse, the mean time between failures as average lifetime durations. Kaleva [51] has shown that when the performance of components in a coherent system is represented by fuzzy numbers, the fuzzy performance of the total system is determined by the structure function in its fuzzified form, as the Extension Principle would suggest. This proof, however, is restricted to the possibility forms found in FN type fuzzy variables since it is based on the Sup-Min operation that only addresses one dimension of possibility (Y-possibility). It does not apply to systems where component lifetime estimates are possibilistic (PN) since the extra U dimension (X-possibility) affects uncertainty management in structural synthesis. The effect of possibilistic failure likelihoods, which represent epistemically feasible likelihoods, on SFS for the basic series and parallel systems will next be described.

#### 7.3.1 Parallel Systems

Components in a parallel system may all be active and online as with designed redundancy, or some may be inactive and off-line as back-up units to be switched on as required. Survival and failure likelihoods for both these types of parallel systems are computed from combinations of possible joint event occurrence in time, which would cause system failure. Only the more common active partial redundancy systems will be discussed below since they illustrate the fundamental joint event in time computation, which is also the basis of the combinatorial calculations in the standby-systems.

Active Parallel Systems -- (one component of “n” required):

When a component fails in this type of redundant system the demand passes to any other available component and system failure only occurs when all components are down at the same time (joint occurrence). This is a time-based definition of failure and can be computed using the AND connective in the extended multiplication form for both PN and FN variables. This is because the joint event definition imputes a chance interpretation (relative frequency) even for possibilistic likelihoods. In other words, a frequency meaning must also be assigned to an epistemic belief measure (PN).

$$\begin{aligned} \text{And, Fuzzy Failure Likelihood of Component (i)} &= 1 - \text{Fuzzy Reliability of Component (i)} \\ &= 1 - \tilde{R\acute{e}l}(i) \end{aligned}$$

$$\begin{aligned} \text{Then, Fuzzy Failure Likelihood of System} &= \prod_{i=1}^n \text{Fuzzy Failure Likelihood of Component (i)} \\ &= \prod_{i=1}^n (1 - \tilde{R\acute{e}l}(i)) \end{aligned} \quad (27)$$

$$\tilde{R\acute{e}l}(\text{System}) = 1 - \prod_{i=1}^n (1 - \tilde{R\acute{e}l}(i)) \quad (28)$$

When “k” components of “n” are required the same simultaneity condition (joint occurrence) applies for failure, and reliability is again estimated by the combinatorial probability formulae in their extended fuzzy forms.

### 7.3.2 Series Systems

For coherent physical systems in which every component contributes to the desired level of system performance, the total system performance is a function of component performance. For such systems, and all forms of U, total system uncertainty increases with each new component added. So if the component failure events are independent, the summation of component U then yields the total or maximum system uncertainty associated with the collection of components.

For a component, let the fuzzy expected number of failure events per unit time be the failure rate ( $\tilde{\lambda}$ ) and the fuzzy expected lifetime be the Mean Time Between Failures (MTBF) =  $1/\tilde{\lambda}$ . The likelihood of finding the component alive or dead at any time is then determined from these fuzzy lifetime estimates.

For a coherent Series system, failure of any component results in system failure.

$$\text{Then, } \tilde{\lambda}(\text{system}) = \sum \tilde{\lambda} \text{ Component (i)} \quad (29)$$

This result is independent of lifetime distributions of components and failure rates need not be constant since the expected probability of failure over an interval can be calculated from any time varying failure rate. Furthermore, when some components are replaced rather than repaired, the Mean-Times-Between-Repair for multi-repaired components can also be combined with the Mean-Times-To-Failure for the replaced items to estimate MTBF(System). Series system reliability estimation for FN and PN lifetime estimates now follows.

*When Lifetime Estimates are FN Fuzzy variables :*

When FN are induced, the system reliability estimate computed from them is the fuzzy probabilistic likelihood (i.e. chance) of finding the system alive at any time (t). (For multiple components, reliability represents the proportion still alive at "t".) The fuzzified version of the normal exponential reliability estimate for components is applied which is based on the Binomial occurrence of events.

$$\begin{aligned}\text{System Performance} &= \text{Fuzzy Reliability of System} \\ &= \text{Fuzzy Probability System is Alive}\end{aligned}$$

$$\begin{aligned}\text{and} \quad \text{R}\tilde{\text{el}}(\text{System}) &= \prod_{i=1}^n (\text{R}\tilde{\text{el}}(i)) \\ &= \prod_{i=1}^n e^{(-\tilde{\lambda}_i t)}\end{aligned}\tag{30}$$

$$= e^{(-n\tilde{\lambda} t)}\tag{31}$$

for n identical components with constant failure rate  $\tilde{\lambda}$ .

When TPD  $(a,b,c)$  are used for practical purposes to represent the fuzzy constant failure rate, this exponential formula is evaluated using an approximation:  $(e^{-an}, e^{-bn}, e^{-cn})$ . Kaufmann and Gupta [52] have demonstrated that for the usually small values of  $(a,b,c)$ , the error introduced by the fuzzy exponential approximation is very small. Consequently, the extra U introduced into FN Reliability computations by this approximation for the fuzzy exponential formula may be neglected. The size of this approximation error for a large TPD failure rate is also shown to be small in Appendix B.

*When Lifetime Estimates are PN Fuzzy variables :*

In this case, the presence of X-Possibility in the component lifetime estimates means that the estimate for it being alive at time (t) represents an epistemic likelihood [0,1] between no belief and total belief (but not certainty). This type of Reliability estimate then differs from the aleatory relative frequency interpretation of the FN derived probabilistic likelihood [0,1], which defines a level of belief between no chance and complete certainty. On the other hand, the epistemic PN likelihood estimate indicates the degree of likelihood that *can* (not will) exist at time "t".

Nevertheless, for a coherent *Series* system, where all parts have an effect, the possibilistic system failure expectation is still the sum of the PN component failure expectations from monotonicity implied in a coherent system.

$$\begin{aligned}\text{System (Failure Rate)} &= \text{Possibility (System Fails in Unit Interval)} \\ \text{and} \quad \tilde{\lambda} \text{ (system)} &= \sum \tilde{\lambda} \text{ Component (i)}\end{aligned}\tag{32}$$

But because the effect of the aggregate X-Possibility in  $\tilde{\lambda}$  (system) determines that the composite likelihood variable is an epistemic possibility rather than a fuzzy probability, relative frequency based laws are now *not* applicable. Thus, neither the reliability of a



component, nor of the system, can be estimated from the exponential expression (13) which is based on multiplicative probabilities. Instead, possibilistic system reliability based on the likelihood of independent events must *be the degree of belief, or sureness, that none of those events occur*. And this estimate must include the full amount of U that is embedded in all the component information. Such an epistemic estimate of system reliability (or non-failure) can be derived from an estimate of the possibility of system failure,  $\tilde{\lambda}$  (system), because in possibility theory [25] the complement of a failure event possibility (i.e. of being dead) represents the degree of sureness (Necessity) that the event does not occur (i.e. of being alive). Then,

$$\begin{aligned}
 \text{Possibilistic System Performance} &= \text{Possibilistic Reliability of System} \\
 &= \text{Degree of sureness that System is alive} \\
 &= \text{Necessity (System is alive)} \\
 &= 1 - \text{Possibility (System is Dead)} \\
 &= 1 - \tilde{\lambda} \text{ (System)} \\
 &= 1 - \sum \tilde{\lambda} \text{ (component i)} \quad (33)
 \end{aligned}$$

Thus, Series System Possibilistic Reliability as a function of time is determined as follows.

System of One Component:

$$\begin{aligned}
 \text{Possibility(System is Dead)} &= \tilde{\lambda} t & \text{for } \tilde{\lambda} t < 1 \\
 &= 1 & \text{for } \tilde{\lambda} t \geq 1
 \end{aligned}$$

System of “n” Equal Components:

$$\begin{aligned}
 \text{Possibility(System is Dead)} &= n \tilde{\lambda} t & \text{for } n \tilde{\lambda} t < 1 \\
 &= 1 & \text{for } n \tilde{\lambda} t \geq 1
 \end{aligned}$$

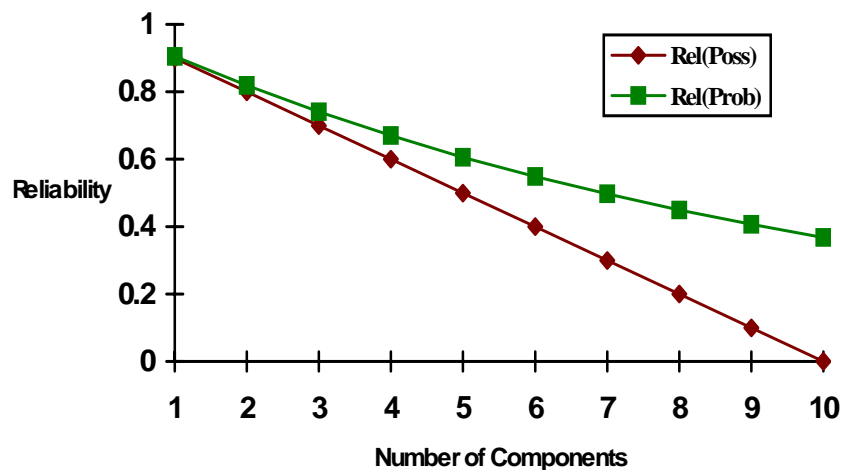
Thus Reliability of n components over time :

$$\begin{aligned}
 R(t) &= 1 - n \tilde{\lambda} t & \text{for } n \tilde{\lambda} t < 1 \\
 &= 0 & \text{for } n \tilde{\lambda} t \geq 1
 \end{aligned}$$

So for a series system with identical components and constant failure rates, there is a linear decrease in possibilistic reliability (PN) with time or number of components. This contrasts to the exponential decrease for fuzzy probabilistic (FN), or crisp probabilistic reliability defined as success frequency across number of demands. These two different reliability functions are illustrated in Figure 16 where it can be noted that:

$$\begin{array}{lll}
 \text{Possibilistic Reliability} & \leq & \text{Probabilistic Reliability} \\
 \text{or Possibility System is dead} & \geq & \text{Probability System is dead}
 \end{array}$$

Importantly, this accords with the Consistency Principle [117] which states that a possibility estimate of an event occurrence should be  $\geq$  a probabilistic estimate for the same event.



**Figure 16:** Series System Reliability - Probabilistic and Possibilistic Comparison

### 7.3.3 Coherent Systems and Structure Function Synthesis

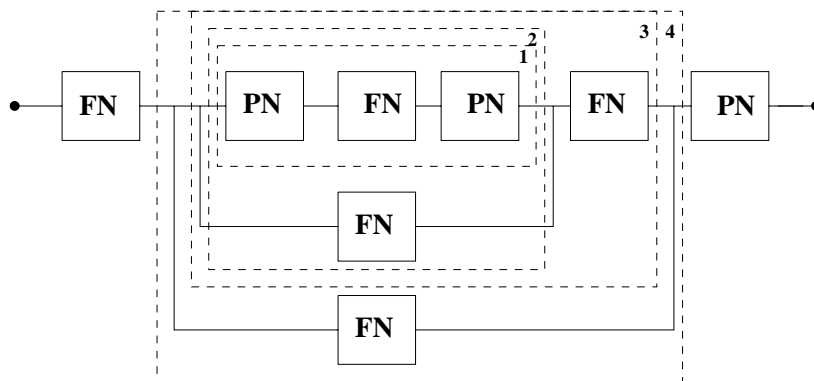
The integrated functioning of components in coherent systems means that every component has a purpose and thus makes a contribution to the overall system objective. If a component were not needed it would not be there, so each component also contributes some  $U$  towards the achievement of the system objective. Some characteristics of coherent systems are that the system fails when all components fail, and that the transition of a component from a failed state to a good state cannot cause system failure. The following metaphor is presented to illustrate that in *physical* systems functional coherency and monotonicity result from the *system structure* which defines the relationships of the parts, and that it is irrelevant whether component reliability estimates are probabilistic or possibilistic. However, this metaphor is not applicable to non-coherent *logical* systems as may occur in fault tree analysis of systems.

Let a house represent a coherent system and let glass windows of various sizes represent components. The house is located in a freezing environment with lots of snow and very strong winds. There is also a small heater in the house to keep a man inside alive. The house is a coherent system because the man who lives in it needs natural light and protection from the freezing temperatures and the house provides the framework that allows the windows to satisfy these needs. However, there are many flying branches which can break the windows and the more windows there are, the more possible it is that heat will escape from a broken window (as increased vulnerability). Heat escape represents system failure and is clearly a function of the number of windows, i.e. increasing the number of windows increases the  $U$  of system failure. If the windows were unconnected and simply lying about outside on the snow then the maximum size present would indicate the maximum possibility of a *component* breakage, but not of system failure, because in this case they are not connected. (In Section 7.7 the possibilistic reliability results of Cai [12] and Misra [68] will be presented which demonstrate this type of extremum evaluation.) Furthermore, leading off the main room of the house is a passage with no windows, and off this passage are a number of small rooms with one window each and special doors that remain open only if someone is inside.

In this analogy, the main room is effectively a series system of window components and the passage system is an active parallel system of windows. Increasing the number of windows in the main room increases the likelihood of death (vulnerability) while increasing the number of rooms with one window off the passage decreases the likelihood of death. In a similar manner, the number of components in physical systems determines the likelihood of system failure regardless of the type of measure (probability or possibility) used to describe each component's reliability. This analogy parallels the monotonicity qualities of a functionally coherent system.

#### 7.3.4 Complex Systems

From the previous results it can be seen that reliability synthesis of a complex (physical) system of series and parallel subsystems, will only be affected if there exists anywhere two or more PN fuzzy variables in effective *series* (since computations for parallel systems with PN are unaffected). It also follows that when PN are in series with FN, each type should be separately grouped and combined before finally combining the reliability estimates of each group. The final variable then becomes a PN by the Principle of Maximum Uncertainty (PMxU). These results for possibilistic variables in series or parallel systems differ from those of other fuzzy reliability authors who apply the Extension principle and set theoretic operations to possibility distribution boundaries. Various fuzzy reliability results of other authors will be summarised in Section 7.7 and this brief introduction to SFS with hybrid U information has demonstrated an alternative approach to soft reliability synthesis. Figure 17 illustrates the stages of reliability synthesis (in dotted boundaries) for a complex hybrid uncertainty system where the computation at each stage is a simple series or parallel reduction. The next example that follows in Section 7.8 compares the results of the proposed procedures for SFS, with those of other fuzzy reliability authors, as well as with traditional probabilistic methods.



**Figure 17:** Hybrid Reliability Synthesis

## 7.4 Fault Tree Analysis

The effects of vagueness (PN) on fault tree analysis (FTA) will now be investigated. FTA is top-down prediction of operational reliability, whereas structure functions are used for bottom-up synthesis of system reliability from component failure likelihoods. FTA is considered top-down since it decomposes the system into subsystems for detailed failure mode and effects analysis. The likelihood of each subsystem failure is then hierarchically evaluated using AND/OR gates and other logical operations (as EXOR) to combine the potential fault and error *events* that can cause it. Thus system reliability is determined from the combination of all potential faults and errors. FTA is widely applied to man-machine systems, and human errors as deviations from standard operating procedures can be included alongside hardware faults. As outlined in Section 7.2 the manner by which the basic event likelihoods are derived determines the nature of those fuzzy estimates which can then influence the implementation of a gate operator. When the likelihood estimates represent fuzzy probabilities of fault/error occurrence they are FN, and when they represent the fuzzy possibility of fault/error occurrence they are PN, which also includes subjective probabilities. The effects of FN and PN on FTA gate implementation are as follows.

### *AND Gate:*

This Gate operator represents simultaneity or joint occurrence of events which is a time-based frequency concept. Thus, a frequency interpretation must also be ascribed to the PN (0.8 of total belief = an occurrence 0.8 percent of time) and the joint event likelihood is evaluated by the extended product of the relative frequencies. So this Gate, as in SFS, has the same implementation for FN or PN likelihoods.

### *OR Gate:*

This Gate operator represents the occurrence of any event from among a set of potential events. For example the FTA for a back-up pump may contain one event for "no power delivered" with three sub-events connected by an OR Gate: switch off, line break, or power supply failure. The OR gate implies "union" which is the sum of the likelihoods minus their duplicated overlaps where joint events can occur. The most direct way to compute the probabilistic likelihood  $\tilde{P}(\text{OR})$  is by using the multiplicity of probabilities for joint occurrence of no failure events, and then taking the compliment which indicates the probability of one or more events.

*For FN Elemental Event Likelihoods:*

$$\tilde{P}(\text{OR}) = 1 - (1 - \tilde{P}_1)(1 - \tilde{P}_2)(1 - \tilde{P}_3) \dots (1 - \tilde{P}_n) \quad (34)$$

where  $\tilde{P}_n$  is the likelihood(probability) of fault n.

*For PN Elemental Event Likelihoods:*

In this case, multiplicity of measures for joint events does not hold by the PMxU because of the X-possibility present. Simultaneity of no events is thus not implied and the aggregate likelihood possibility for  $\tilde{P}(\text{OR})$  is simply the sum of the elemental likelihoods. While this equates to the FN evaluation for mutually exclusive events, for PN it is an unconditional

evaluation which yields higher values than the probabilistic OR gate. It should also be noted that this possibilistic likelihood of system being functional, does not mean the *potential* of the system to be functional, which would simply be the Max of the component Necessity measures to be alive.

So for PN estimates:

$$\tilde{P}(\text{OR}) = \Sigma \tilde{P}_n \quad (35)$$

where  $\tilde{P}_n$  is the likelihood (possibility) of Fault n.

In cases where the event likelihoods are all very small, the FN and PN computations are approximately equal, as well as for mutually exclusive events of course. Another OR gate variant is the Exclusive OR gate (EXOR) which is used to evaluate the likelihood of one and only one occurrence from a set of mutually-exclusive events such as multi-state failures. The EXOR operator introduces non-coherent aspects and requires a different treatment for FN and PN variables. For FN, the EXOR is evaluated by replacing it with suitable AND/OR combinations to model the conditionality implied by mutually exclusive events. In this way, the model is reduced to a coherent structure with only AND/OR gates. For PN, it would seem that the extended Max extremum operator is most appropriate to evaluate EXOR. This extremum computation selects a single maximum likelihood value across all levels of belief and the result is a composite of all sets representing their extreme fuzzy likelihood value. Real-world fault trees must also include other complications such as dependent or repeated events, but these are beyond the scope of this introductory treatment of the subject. However, one simple type of dependency effect that can induce a PN in SFS computations will be discussed in Section 7.6.

## 7.5 Monotonicity in Reliability Analysis

The effect of the previous results for the two types of fuzzy variable on the monotonicity of system behaviour will now be summarised. In both SFS and coherent FTA, monotonicity only applies to the basic system elements (Series/Parallel systems or AND/OR gates) since the effect on the total system of any change depends on the type of element that is incrementally changed. The forms of monotonicity in reliability analysis are as follows.

SFS:

For increasing number of components in series systems,  $\tilde{\lambda}$  (System) increases arithmetically for both FN and PN estimates while reliability decreases exponentially for FN and by linear decrease for PN. For increasing number of components in active parallel systems,  $\tilde{\lambda}$  (System) and reliability decreases geometrically for both FN and PN. For increases in component failure likelihoods, reliability decreases exponentially for FN and linearly for PN.

FTA:

With an increasing number of faults or failure likelihoods, under the AND Gate for FN or PN, the event likelihood decreases geometrically while reliability increases likewise. For the OR Gate with FN, the event likelihood increases and the reliability decreases geometrically. For OR Gate with PN, the changes are linear.

So overall, monotonicity is preserved in coherent systems for both FN and PN variables in soft reliability analysis. However, various factors such as interdependencies can induce non-coherent behaviour in systems and this must be removed from the model before the previous procedures can be applied.

## 7.6 Dependent Failures

There are a variety of approaches to system modelling of dependency, each for a different type of real-world dependency. In abstract decision models, for example, dependency usually means a lack of independency due to variable synergies or some type of interactivity. One approach to modelling this type of dependency is to use tools such as non-additive fuzzy measures and fuzzy integrals. In reliability modelling of physical systems, dependency usually means a behavioural link between components, such as when the failure of one component affects the lifetimes of others. However, there can be at least three types of behavioural links between physical subsystems. The first type refers to conditional events as in parallel standby systems where there are idle back-up components. Here the functioning of the idle components is *conditional* on the failure of the primary component and some form of conditional computations must be used to estimate system performance. The second type of dependency is demonstrated in active parallel systems where all components are on-line but the lifetime of each component can be affected by the failure frequency of other components. This can occur because failure increases the stress or *load* on other components which consequently decreases their lifetimes. This effect is prevalent in electronic and electro-mechanical systems. Another example of this type of dependency is in series systems where a shutdown/start-up effect caused by the failure of any component can introduce sporadic stress that increases the failure rates of the other functioning components. A third type of dependency in physical systems is "common-cause" failure where the failure of one component or aspect can affect many others. Examples of common causes are power failure which can affect many subsystems, or opening an email attachment with a virus that can spread across a system. Usually, common-causes can be incorporated into the design of the fault-tree as extra events, but the PN implications for these procedures are also beyond the scope of this work. What will next be investigated is the second type of load dependency which affects SFS for systems where there are interrelated failure events. To investigate the PN implications for this type of system, two modes of reliability analysis should be distinguished.

*Dynamic Reliability Estimation*

( *A Posteriori* )

In this type of SFS the objective is to estimate the current reliability of a functioning (dynamic) system. It is based upon some updating procedure which uses recent behavioural evidence to estimate component lifetimes and then synthesise the system reliability from the structure function. Since the evidence reflects the net effects of all dependencies, dynamic updating automatically accommodates dependencies in the estimate and the presence of dependent

faults requires no special treatment. However, a different kind of reliability analysis may also be required as follows.

### *Reliability Prediction*

(A Priori)

In this case, the objective is to estimate the reliability of an hypothetical system (which is not in existence). This may be at the design stage or be a prototype. The inputs are stand-alone component reliabilities which may be obtained from industrial standards or database, and component interdependency strengths. A method will now be presented which is pragmatic, although simplistic, and uses expert experience to approximate the load dependencies between components. These are then combined with the stand-alone component lifetime estimates to estimate the hypothetical reliability of the system. The proposed method for incorporating the effects of dependent faults will be illustrated using a simple active parallel system with three identical components (A,B,C) in parallel. Suppose an expert estimates that the failure of any component will increase the failure rates of the other components by 10%. Because this is a *subjective* estimate, all component failure rates must be then treated as PN, so the component failure likelihood within time T is  $\lambda T$  (i.e. not  $1 - e^{-\lambda T}$ ). The following example estimates the system reliability at  $T = 100$  for equal component failure rates of  $\lambda = 0.003$ . An effective failure rate ( $\lambda_{\text{Eff}}$ ) for each component is first computed to capture the interdependency effect. This is considered to be the stand-alone failure rate plus the percentage increase due to the other components failing.

With dependency between components and  $\lambda$  is an induced PN:

$$\begin{aligned}\lambda_{\text{Eff}}' &= \lambda_A + 0.1 \lambda_B + 0.1 \lambda_C \\ &= 1.2 \lambda \text{ (for identical } \lambda \text{)}\end{aligned}$$

$$\begin{aligned}R_{\text{SYS}} &= 1 - (1.2 \lambda T)^3 \\ &= 1 - (1.2 \times 0.3)^3 \\ &= 0.9534\end{aligned}$$

With *no* interdependency and all  $\lambda$  are (crisp) FN:

$$\begin{aligned}R_{\text{SYS}} &= 1 - (1 - e^{-0.300})^3 \\ &= 1 - 0.0174 \\ &= 0.9826\end{aligned}$$

With *no* interdependency and all  $\lambda$  are (crisp) PN:

$$\begin{aligned}
 R_{\text{SYS}} &= 1 - \prod_1^n (1 - R_i) \\
 &= 1 - (0.3)^3 \\
 &= 1 - 0.027 \\
 &= 0.973
 \end{aligned}$$

This example demonstrates the decrease in reliability when interdependency is present, and also that FN reliability is higher than PN reliability. But what is even more significant is that the system reliability (and failure likelihood) must now be an epistemic PN because of the subjective estimate of the dependency factor. Consequently, all higher or further system computations must treat this system performance estimate as a PN which can further magnify the numerical increase shown above. This method can similarly be applied to series system synthesis.

## 7.7 Soft Reliability Literature Survey

A brief survey of the main trends in the literature on soft reliability is now presented for comparative purposes. Most of the literature discussed below is summarised in a recent volume [73] of collected papers on fuzzy reliability analysis.

### 7.7.1 Fuzzy Fault Tree Analysis

There are two stages to the evaluation of a fault tree: the first which is the qualitative identification of cut-sets, and the second which is the quantitative evaluation of the combined event likelihoods. In large systems, computational complexity exists at both stages and several authors have developed algorithms for these aspects of FTA with fuzzy likelihoods. However, the following selection does not include this category of literature pertaining to algorithmic aspects, but rather focuses on the conceptual models used and fundamental theoretical aspects. Accordingly, two theoretical categories are identified based on the use of *probability* or *possibility* estimates for event likelihoods.

The first category concerns models using fuzzy fault/error *probabilities*. This approach models likelihood estimates as fuzzy probabilities and uses extended probability versions of the gate operators. Although there are some rather fine modelling distinctions between the authors, overall, the approach and results are equivalent to those presented in this paper for FN representation of fuzzy probabilities. Singer [87] provides one example of this approach among many that exist in the literature.

The second category concerns models using fault/error *possibilities* induced when estimates of event likelihoods are linguistic terms such as "highly unlikely", or by subjective probabilities (considered here as possibilities), or simply by feasibility estimates. Investigations using such



possibility estimates implement the logical gate operators either by *probability-like* operators, or by *extremum* operators as extended Min/Max, which effectively implies a non-coherent system of possibilities.

Onisawa [74] provides an example of FTA using *possibility* likelihoods with *probability-like* operators. He points out that the extremely small failure probabilities used in statistical fault tree analysis are seldom based on adequate statistical evidence and suggests that expert opinion be used acquired through linguistic estimates such as "very unlikely". From these input ratings a log function is then used to model the very small error possibilities and subsequently a parametric possibility distribution is developed. This distribution includes a parameter similar to a safety-factor which controls the fuzziness of the estimate. These error possibility distributions are the computational elements that are input to the AND/OR gates using compensatory Dombi operators, which are approximately compatible with the probability operators of algebraic sum and product. However, no reasons for choosing the Dombi operator can be located by this author and it does seem rather hard to justify. So the choice of this operator has introduced Level 3 macrostructure U into this method of Onisawa.

Another approach to FTA with *possibility* likelihoods is demonstrated by Misra and Weber [68,69] who develop possibility estimates using the extremum operator (extended Min and Max). This effectively dissociates the basic elements and imputes non-coherent behaviour to the logical system in the FTA. (Consequently, this method should not be used for SFS because physical systems normally exhibit coherent behaviour as illustrated in the previous "house" analogy. Nevertheless, the use of the extremum operator in SFS will be demonstrated in the example in Section 7.8. for comparison purposes.) As explained in the previous FTA discussion, the extremum acts as an EXOR gate for *mutually exclusive* events when the event likelihoods are possibilities, but when a *coherent* OR gate is required in a model, the extremum is not suitable because it drops uncertainty information concerning some events.

The above approaches to estimating event *possibilities* in FTA diverge from the methods of this author in two ways. One is their emphasis on the error possibility *distribution* itself. This contrasts with the emphasis in this report on the *uncertainty content* in the possibility distribution, as well as the adoption of a family of simple fuzzy sets (TPD or TpPD) for representing the uncertainty in the elemental estimates. A further important point of divergence is the manner of implementing the Union of event likelihoods when higher-order U is present.

### 7.7.2 Fuzzy Structure Function Synthesis

Two categories of literature using probability and possibility estimates are also identified for Structure Function Synthesis (SFS).

SFS for coherent systems with fuzzy probabilities has been investigated by Kaleva [51] who proved that when the performance measures of components are fuzzy numbers (FN), the performance of the coherent system is estimated by the extended structure function. However, as previously stated, that proof rests on Sup-Min mappings which only evaluate the Y-possibility dimension as exist in FN.

For Structure Function Synthesis with possibilistic likelihoods (PN), there are two principal streams of literature. One of these is from Cappelle and Kerre [13] who apply lattice theory for soft reliability estimation of systems. Some results of Cappelle and Kerre are:

- Possibilistic Reliability is independent of structure function
- There is no unique relationship between possibilistic reliability and component lifetime.
- Possibilistic System Reliability does not vary monotonically with component additions to basic structural elements.
- For Series Systems:  $\text{Rel}(\text{System}) \leq \text{lowest Rel}(\text{component})$
- For Parallel Systems:  $\text{Rel}(\text{System}) \geq \text{highest Rel}(\text{component})$

It is shown in the example that follows that the implementation of these inequalities can only produce limited results regarding system reliability estimates.

The other stream of possibilistic reliability literature is from C.Y. Cai. [9,10,11,12] and some of his results parallel those of the above authors. Cai classifies general fuzzy reliability as follows.

- Two main classes of fuzziness which induce soft reliability estimates are identified -
  - \* Fuzzy failure *events* as in components with multi-state failure
  - \* Fuzziness as induced possibility due to *lack of information* ,  
in [9] "probability stems from sample generality whereas possibility characterizes sample particularity".
- Based on this division of fuzziness Cai identifies four potential combinations -
  - \* PROBIST - Probability information on Binary state events
  - \* PROFUST - Probability information on Fuzzy state events
  - \* POSBIST - Possibilistic information (statistically insufficient) on Binary events
  - \* POSFUST - Possibilistic information on Fuzzy state events.

The first category can be addressed by normal statistical methods but the other three are soft reliability variants. Cai investigates PROFUST by using state-space Markovian modelling for a simple system and develops some general results for the POSBIST category. He does not investigate the POSFUST category.

- Some of Cai's results are:
  - \* PROFUST -  
As with applications of Markovian methods, time based functions are derived for reliability of the simple example system from which steady-state behaviour is extrapolated.
  - \* POSBIST - (For coherent systems)
    1. There is no unique relationship between reliability and lifetimes (or failure rates). However, the ordinal relationship of component lifetimes defines the ordinal relationship of their reliabilities (as one could well expect). [10,p163]

2. The possibilistic reliability function decreases monotonically with time but not with component additions to basic structural elements. [10, p156]

(This is a surprising inference which is due to Max/Min operations on the separate component distributions. )

3. Reliability boundaries for any system can be determined from extreme values of the component (i) reliabilities: [10, p164-7]

$$\text{Rel (Series System)} = \text{Min ( Rel (i) )}$$

$$\text{Rel (Parallel System)} = \text{Max ( Rel (i) )}$$

$$\text{Min ( Rel (i) ) } \leq \text{Rel (Complex System)} \leq \text{Max ( Rel (i) )}$$

4. Limits for System Lifetime (L) are thus determined from component lifetime limits: [10, p167]

$$L \text{ (Series System)} = \text{Min ( L(i) )}$$

$$L \text{ (Parallel System)} = \text{Max ( L(i) )}$$

Thus, for all systems with varying numbers of identical components with lifetimes L,  $L \text{ (System)} = L$

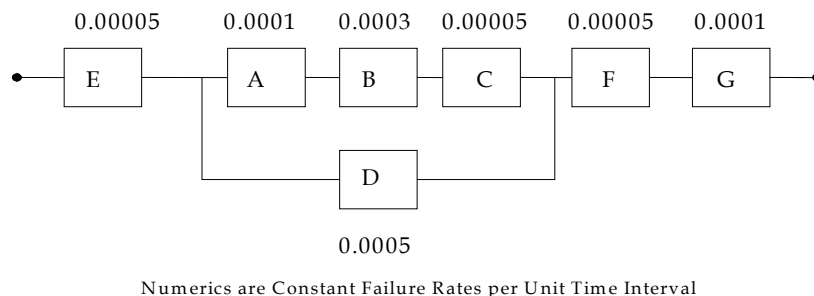
A similarity between Cai's approach and the approach of this author is the separation of U within pieces of data, from that U introduced by the multiplicity of data. However, a major difference is that Cai considers that any fuzziness within these two types of U can be represented by a single dimension of possibility, whereas the procedures of this author allow for the potential existence of two dimensions of possibility within each division of U. The difference between these viewpoints results in a different definition for a possibilistic variable. Another major difference is with the implementation of conjunction and disjunction of events in a system. Cai applies Max and Min operations on the set of individual event likelihoods, whereas the implementation of possibilistic event conjunction and disjunction in this report is regulated by applying U management principles to the total U in the whole system. Regarding Cai's PROFUST method, it could be said that Markovian analysis is not a general solution for reliability analysis where fuzzy failure events are present, since it only applies to one type of non-binary failure that is due to redundancy as with a number of components in parallel operation. In such parallel systems, redundancy allows operation to continue when a component fails and the system deteriorates step-wise until the complete failure state when all components have failed. Furthermore, the Markovian method for more complex networks with non-binary failure events, is severely limited by computational complexity. Regarding Cai's POSBIST fuzzy reliability class, Cai's results are shown in the following example to have practical limitations, as well as violating the PMxU for coherent systems because the total system U does not increase with increasing components. Both of the previous comments also apply to the results of Cappelle and Kerre.

Overall for possibilistic SFS, the results of the above authors are divergent from those proposed by this author because they are founded on Sup-Min operations on the *boundaries* of possibility distributions with only on a single dimension of possibility. Consequently, the results do not apply to data with higher forms of fuzzy uncertainty. Another approach to possibilistic SFS could be the application of the extremum operator for conjunction and disjunction of failure events, as used by Misra for FTA. However, this would impute non-coherent behaviour to the physical system by introducing a type of interdependency between

events since the occurrence of any event excludes the likelihood of any other event. While this may be valid in block diagrams which are transpositions of logical fault tree relationships, it is not suitable for block diagrams that represent the *functional* relationships of physical subsystems with independent and coherent behaviour.

## 7.8 Comparative Example for Structure Function Synthesis

A simple system (Figure 18) will be used to demonstrate different methods for computing system reliability estimates using the SFS technique. In reliability analysis, numerical magnitudes for failure event likelihoods are normally extremely small and the differences between the following results may appear to be insignificant. However, they would be magnified greatly as system complexity increases and/or mission time increases. The example in Figure 18 may be considered to be a large Command and Control system where the subsystems are discrete C2 cells and the basic unit of time is a 6 hour shift. For computational simplicity, only crisp failure likelihoods are used which may be considered as the single modal value of a fuzzy TPD estimate. But if full TPD were used, the left and right values would also exhibit a similar divergence in values to those shown. The subsystems are considered to be independent and have constant failure rates which imply as-good-as-new restoration. For a mission time of 1000 (or 250 days) the following results highlight the divergence between the estimates of the different methods.



**Figure 18:** *An Example System*

For the example system in Figure 18, the following computations on soft reliability synthesis demonstrate the methods of this report for all component measures FN, or all PN, as well as for the fuzzy probability methods of K.Y.Cai [10,12], Cappelle/Kerre [13], and Misra [68,69].

Computations for the Method of this Report:

If all subsystems are FN:

Since crisp FN are used for simplicity, the following computations are the same as for traditional probability theory.

$$\lambda_{A,B,C} = 0.0001 + 0.0003 + 0.00005 = 0.00045$$

Let system lifetime be the expected lifetime as estimated by the Mean Time Between Failures (MTBF) . Then, for the active parallel system (A,B,C)D:

$$\begin{aligned} \text{MTBF} &= \int_0^{\infty} R(t) dt \\ &= \int_0^{\infty} \left[ 1 - (1 - e^{-0.00045t})(1 - e^{-0.0005t}) \right] dt \end{aligned}$$

For active parallel systems with 2 components then  $\text{MTBF}_p$  :

$$\begin{aligned} &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2} \\ &= \frac{1}{0.00045} + \frac{1}{0.0005} - \frac{1}{0.00095} \\ &= 3169 \end{aligned}$$

$$\text{Thus } \lambda_p = \frac{1}{3169} = 0.0003156$$

$$\begin{aligned} \text{So for final series system: } \lambda_{\text{System}} &= 0.00005 + 0.0003156 + 0.00005 + 0.0001 \\ &= 0.0005156 \end{aligned}$$

$$\text{And MTBF (System)} = \frac{1}{\lambda_{\text{System}}} = 1939 \text{ Shifts}$$

(For the case where all components are identical:

$$\text{MTBF}_{\text{PARALLEL}} = \frac{1}{\lambda} \sum_{i=1}^k \frac{1}{i} \quad \text{where } k = \text{number of components in parallel}$$

For example:

$$2 \text{ components and } \lambda = 0.00045, \quad \text{MTBF}_p = \frac{1}{0.00045} \left( 1 + \frac{1}{2} \right) = 3333$$

$$\text{or for } \lambda = 0.0005, \quad \text{MTBF}_p = \frac{1}{0.0005} \left( 1 + \frac{1}{2} \right) = 3000$$

Then, Reliability of example system at T = 1000:

$$R_{A,B,C} = e^{-4.5 \times 10^{-4} \times 10^3} = e^{-0.45} = 0.638$$

$$R_D = e^{-5 \times 10^{-4} \times 10^3} = e^{-0.5} = 0.6065$$

$$R_{(A,B,C,D)} = 1 - (.362)(.3935)$$

$$= 1 - .1424$$

$$= 0.8576$$

$$\text{And } R_{\text{SYSTEM}} = R_E \times R_P \times R_F \times R_G$$

$$= e^{-5 \times 10^{-2}} \times 0.8576 \times e^{-5 \times 10^{-2}} \times e^{-1 \times 10^{-1}}$$

$$= .95123 \times .8576 \times .95123 \times .9048$$

$$= 0.702$$

This probability represents the survival probability for the system when it has aged to T=1000, and the complement is the failure probability at that age. Thus, the value indicates that after 250 days (or 1000 time units) there is 0.7 chance of being alive. For a constant load, this is also the instantaneous Availability estimate as the expected proportion of time fully operational over a small time interval (say a few days). However, the reliability value does not indicate the expected proportion of time operational over a longer time window, the interval availability estimator being required for that purpose. For military type shifts of 12 hours, 1000 periods is quite a long interval resulting in this fairly low reliability value. More realistically, 500 periods (or 250 days) would result in a reliability value somewhat higher than the above value.

The above calculation also illustrates that the system failure rate ( $\lambda_{\text{System}}$ ) is not constant simply because the component failure rates are constant. Rather, the FN system failure rate of 0.0005156 represents an average value over the time interval  $[0, \infty]$  derived from the MTTF *over the same interval*. If this were a constant failure rate, the reliability at T=1000 would be 0.597 ( $e^{-0.0005156T}$ ) rather than 0.702.

*If all subsystems are PN:*

$$\text{Again, } \lambda_{A,B,C} = 0.0001 + 0.0003 + .0.00005 = 0.00045$$

But for PN, the possibility of failure of each arm of the parallel system (A,B,C,D) in time  $t$  is  $\lambda_1 t$ . So for both arms, the possibility of joint failure within time  $t$  with no restoration is  $(\lambda_1 t) \times (\lambda_2 t)$  and this joint possibility must be  $\leq 1$ , yielding a time limit  $T^*$  for certain failure at the equality.

$$\begin{aligned} T^* &= \left( \frac{1}{\lambda_1 \lambda_2} \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{0.00045 \times 0.0005} \right)^{\frac{1}{2}} \\ &= 2108 \end{aligned}$$

Then for A,B,C,D:

$$Rel_{para} = 1 - Poss \text{ (failure at } t \text{)}$$

$$= 1 - \lambda_1 \lambda_2 t^2$$

$$\text{So } MTBF_{Para} = \int_0^{T^*} R(t) dt$$

$$= \int_0^{T^*} [1 - \lambda_1 \lambda_2 t^2] dt$$

$$= [t]_0^{T^*} - \frac{\lambda_1 \lambda_2}{3} [t^3]_0^{T^*}$$

$$= 2108 - \frac{22.5 \times 10^{-8}}{3} (9.3672 \times 10^9)$$

$$= 2108 - 702$$

$$= 1406$$

$$\text{Thus } \lambda_{Para} = 0.00071$$

$$\text{and } \lambda_{System} = 0.00005 + 0.00071 + 0.00005 + 0.00001$$

$$= 0.00091$$

$$\text{and } MTBF_{SYSTEM} = 1099 \text{ TimeUnits.}$$

For Reliability at  $T = 1000$ ,

$$\text{Poss(Dead)}_{A,B,C} = 0.00045 \times 1000 = 0.45$$

$$R_{A,B,C} = 1 - 0.45 = 0.55$$

$$\text{Poss(Dead)}_D = 0.0005 \times 1000 = 0.50$$

$$\text{Thus Poss(Dead)}_{A,B,C,D} = 0.50 \times 0.45$$

$$= 0.225$$

$$R_{A,B,C,D} = 1 - \text{Poss(Dead)}_{ABCD} = 0.775$$

$$\text{Then Poss(Dead)}_{\text{SYSTEM}} = \text{Poss(Dead)}_E + \text{Poss(Dead)}_{ABCD} + \text{Poss(Dead)}_F + \text{Poss(Dead)}_G$$

$$= 0.05 + 0.225 + 0.05 + 0.1$$

$$= 0.425$$

$$\text{and } R_{\text{SYSTEM}} = 1 - \text{Poss(Dead)}_{\text{SYSTEM}} = 0.575$$

This possibilistic (PN) reliability value based on subjective PN estimates of what can happen is thus 18% lower than the probabilistic reliability value (0.57 : 0.70 ).

Results from the fuzzy set theory methods of other authors now follow for comparison.



**Cai's Method:**

Let Subsystem lifetime	$L$	$=$ MTBF	$= 1 / \lambda$
In series within loop:	$L(ABC)$	$=$ Min ( $L(A), L(B), L(C)$ )	$= 1 / .0003$
In parallel system (ABCD):	$L(ABCD)$	$=$ Max ( $L(ABC), L(D)$ )	$= 1 / .0003$
In Total series system:	$L(\text{System})$	$=$ Min ( $L(E), L(ABCD), L(F), L(G)$ )	$= 1 / .0003 = 3333 \text{ shifts}$

**The Cappelle/Kerre Method:**

By application of the Capelle/Kerre inequalities:

Series System (ABC) :	$\lambda \geq 0.0003$
Parallel System (ABCD):	$\lambda \leq \text{Min}$
	$\leq 0.0005$ for certain because $\lambda(ABC) \geq 0.0003$
Series system ( System )	$\lambda \geq \text{Max}$
	$\geq 0.0001$ for certain

Thus  $\lambda(\text{system}) \geq 0.0001$  and the upper lifetime limit is 10,000 shifts.

**Misra's Method (Extremum Operator for No Joint Events)**

Series ABC :	Max $\lambda$	$= 0.0003$
Parallel system (ABCD)	Min $\lambda$	$= 0.0003$
Series system (System)	Max $\lambda$	$= 0.0003$ .

Thus the extremum operator yields  $\lambda = 0.0003$  or an expected life of 3333 shifts.

Some comments comparing the above results can now be made.

*For the procedures of this author:*

- The extra U in the PN variables results in a lower reliability than fuzzy probability(FN) at any length of operating time.
- The difference between the PN (0.575) and FN (0.702) reliability estimates is significant.
- The expected lifetime value (MTBF) is also significantly lower for PN (1,099) than for FN (1,939).

*Cai's Results:*

- Although Cai says that possibilistic reliability decreases with time, no value can be determined.
- System lifetime is 3333 shifts based on Max and Min operations.

*Capelle / Kerre's Results:*

- No reliability estimate is possible
- The maximum possible lifetime estimate is 10,000 shifts.

*Misra's Extremum Method:*

- No reliability estimate is possible.
- System Lifetime is 3333 shifts (higher than both FN and PN values of this author's procedures because of the absence of joint events).

In this example, the significant difference between the predicted FN and PN reliability values demonstrates that it can be important to acknowledge the effects of the different forms of fuzziness that may be present to ensure that the estimate of operational likelihood does match the real-world (i.e. no big estimation errors). Furthermore, the fact that no likelihood estimate (probability or possibility) can be considered as accurate by their intrinsic nature, requires that all forms of U in the estimate (as embedded in the fuzzy shape of a likelihood distribution) should be carefully managed in computations. To this end, with the aid of the PN concept, the general procedures presented here for soft reliability analysis allow fuzzy estimates to be computed in a manner that fits the specific forms of fuzziness present. For a hybrid U system where fuzzy and non-fuzzy variables are present, the SFS method can also be applied using the standard trapezoidal distribution to represent all types of variables because it can represent singleton, triangular, and rectangular possibility (and probability) distributions as well.

**Table 14:** *Summary of Reliability Computations*

ANALYTICAL METHOD	EXPECTED LIFE (L)	RELIABILITY (1000)
This Author: FN (Probabilistic)	1939	0.702
PN (Possibilistic)	1099	0.575
CAI :	3333	None
CAPELLE/KERRE :	$L \leq 10,000$	None
MISRA:	3333	None

## 7.9 Section Summary

This section has demonstrated the practical application of the proposed methods for hybrid uncertainty management in complex numerical induction models, through the field of system reliability analysis and the techniques of structure function synthesis and fault tree analysis. In light of the poor quality data frequently used in much real-world reliability analysis, it is suggested that the proposed possibilistic procedures provide a more reasonable approach that is also a more conservative approach. The results of the proposed procedures have also been compared with those of some of the principal authors in the field of soft reliability analysis, and the significant divergence of reliability estimates highlighted by means of a complex system reliability example.

## 8. Conclusion

The subject of this report has been the representation and measurement of the hybrid uncertainty forms as may exist in what has been called higher-order uncertainty. The specific meaning that has been assigned to higher-order uncertainty in this report is essentially various fuzzy uncertainty forms beyond crisp numerical or probabilistic likelihood estimates. This also includes different types of fuzzy probabilities when there is an insufficient sample of evidence to derive crisp probability estimates.

The primary thesis of this report is that there can be two distinct types of fuzzy uncertainty induced, ambiguity and vagueness, and that these are not clearly distinguished by the prevailing Fuzzy Set and Possibility Theory concepts. To support this supposition, new uncertainty representations and definitions have been proposed for vagueness, with new measures also defined for both ambiguity and vagueness. The behaviour of these new measures has then been demonstrated using various examples and compared with some other fuzzy uncertainty measures that have been proposed. This comparison has indicated that the behaviour of the new measures appears to be more consistent and reasonable compared to the somewhat erratic and insensitive behaviour of the more traditional fuzzy uncertainty measures. The traditional fuzzy uncertainty measures were also shown to be fundamentally based on the ambiguity concept, as the notion of the cardinality of a set being the aggregate number of elements within the set. As well as failing to represent vagueness the traditional fuzzy uncertainty measures do not provide any information on the relative proportion of uncertainty in the numerical representation. Extensions were then presented for the traditional measures to enable them to address vagueness when it also is present in information. Moreover, the proposed methodology for identifying hybrid uncertainty forms and reducing them to combinations of ambiguity and vagueness was demonstrated in the following applications:

- the identification of appropriate techniques for summarising data sets containing different hybrid uncertainty combinations;
- the development of the Stochastic Fuzzy Number which embeds some statistical information in a fuzzy set when only a sparse data set is available;
- the Composite Uncertainty Interval, being a crisp interval as used in interval mathematics that represents the aggregate of the hybrid uncertainties;
- the management and aggregation of hybrid uncertainty forms in numerical induction models through the examples of fault tree analysis and complex system reliability synthesis.

The conclusion from the investigations of this report is that a careful identification of the uncertainty sources present can enable the presence of any vagueness in information to be identified. Using the proposed methods, this form of uncertainty may then be managed in aggregation computations alongside the more common form of ambiguous uncertainty. With increased fidelity of uncertainty representation and management, more meaningful results should then be derived from computations. And since various types of decision analysis models used in Defence commonly process soft information, using the proposed methods in those numerical induction models should also naturally lead to better decisions.

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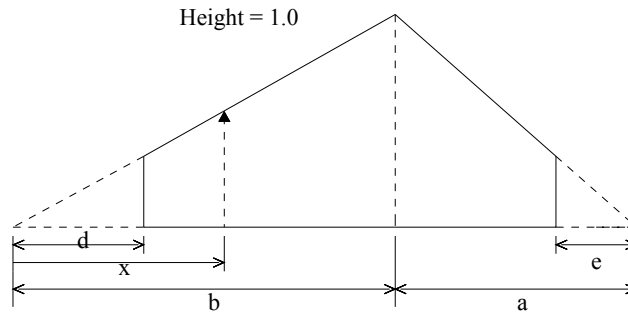
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## Appendix A: Centre of Area Formula for Truncated Triangular Possibility Distributions

A general formula is developed below for computing the Centre of area (COA) for truncated (or non-truncated) triangular possibility distributions. The formula has particular relevance to the Stochastic Fuzzy Quantity (SFQ) proposed in Section 6.3 which is a fuzzy distribution with some stochastic information embedded in it. The SFQ can be used for representing the information contained in a small sample of data, as may pertain to observations of rare (or unexpected) events. The COA of the SFQ can be taken as a measure of typicality for the statistically sparse set for use in subsequent computations. It is the preferred defuzzification method because it conforms to the ordinal nature of a possibility distribution whereas the other common method, the Centroid or Centre of Gravity method, is based on an expression which is meaningless for possibility distributions. As described in Section 6.3, the only forms of SFQ that can occur are complete triangular possibility distributions, left truncated TPD, or trapezoidal with crisp right side and a left slope that may or may not be truncated. However, most commonly, the triangular form would result from the SFQ computation.



**Figure 19:** A Truncated Triangular Possibility Distribution

At any value  $x$  :

$$\text{Left Side Area} = \frac{1}{2}(x-d)\left(\frac{d}{b} + \frac{x}{b}\right)$$

$$\text{Right Side Area} = \frac{1}{2}(b-x)\left(1 + \frac{x}{b}\right) + \frac{1}{2}(a-e)\left(1 + \frac{e}{a}\right)$$

And at COA : Area Left of  $x$  = Area Right of  $x$

$$\text{Thus, } \frac{1}{2}(x-d)\left(\frac{d}{b} + \frac{x}{b}\right) = \frac{1}{2}(b-x)\left(1 + \frac{x}{b}\right) + \frac{1}{2}(a-e)\left(1 + \frac{e}{a}\right)$$

$$(x-d)(x+d) = b\left[(a-e)\left(\frac{a+e}{a}\right) + \frac{(b-x)(b+x)}{b}\right]$$

$$x^2 - d^2 = \frac{b}{a} (a^2 - e^2) + (b^2 - x^2)$$

$$2x^2 = \frac{b}{a} (a^2 - e^2) + b^2 + d^2$$

$$\text{and at COA, } x = \sqrt{\frac{1}{2} \left[ \frac{b}{a} (a^2 - e^2) + b^2 + d^2 \right]}$$

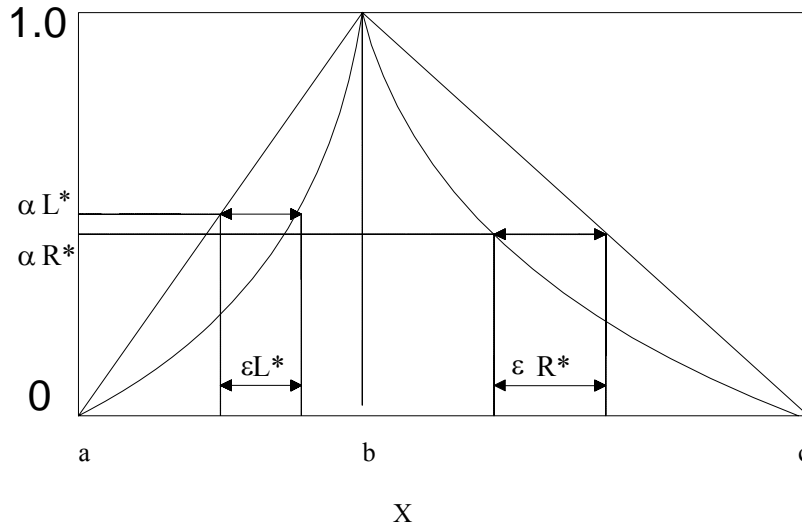
**Notes:**

1. For symmetrical TPD the above formula simply defines the midpoint of the base.
2. For  $a = d = e = 0$ ,  $x = \sqrt{(b^2/2)}$  and for  $b = d = e = 0$ ,  $x = \sqrt{(a^2/2)}$  from right side
3. For the Stochastic Fuzzy Quantity (SFQ) :

$$a = \frac{U-M}{1-\underline{\pi}} \quad \text{and} \quad b = 2(M-L), \quad d = M - 2L, \quad e = 0$$

where the SFQ is defined by  $\{ L, 0.5; M(\text{Median}), 1; U, \underline{\pi} \}$ .

## Appendix B: Fuzzy Exponential Approximation Error for a TPD



**Figure 20:** The Linear Approximation of a Fuzzy Exponential Number

The following equations summarise the method of Kaufmann and Gupta [52] for computing the error (  $\varepsilon$  ) introduced by approximating a non-linear fuzzy exponential number with a linear triangular possibility distribution as used in Section 7.3.

Let  $A = (a, b, c)$  be a triangular possibility distribution

Let  $A' = \exp^A = \exp^{(a,b,c)} = e^{(a,b,c)}$

Interval at any  $\alpha$ :  $A'_\alpha = e^{[a+(b-a)\alpha, c-(c-b)\alpha]}$   
 $= e^{[a+(b-a)\alpha]}, e^{[c-(c-b)\alpha]}$

Let approximation of  $\exp^A = P = e^a, e^b, e^c$

with  $\alpha$ -cuts  $P_\alpha = [e^a + (e^b - e^a)\alpha, e^c - (e^c - e^b)\alpha]$

and maximum Left divergence (  $\varepsilon_L^*$  ) at  $\alpha_L^* = \frac{-a}{b-a} + \frac{1}{b-a} \ln \left( \frac{e^b - e^a}{b-a} \right)$

and maximum Right divergence (  $\varepsilon_R^*$  ) at  $\alpha_R^* = \frac{c}{c-b} + \frac{1}{c-b} \ln \left( \frac{e^c - e^b}{c-b} \right)$

For  $(a, b, c) < 1$ ,  $\varepsilon^*$  are small and can be neglected.

Example: For a TPD(a,b,c) failure intensity

Let  $a=0.01$ ,  $b=0.30$ ,  $c=0.5$  (a fairly large failure intensity)

$$\begin{aligned}\text{Then, } \alpha_R^* &= \frac{c}{c-b} - \frac{1}{c-b} \ln \left( \frac{e^c - e^b}{c-b} \right) \\ &= \frac{.5}{.5-.3} - \frac{1}{.5-.3} \ln \left( \frac{e^{.5} - e^{.3}}{.5-.3} \right) \\ &= 0.495\end{aligned}$$

And  $\varepsilon_R^* = \text{Exact value} - \text{Approx. value}$

$$\begin{aligned}&= e^{c-(c-b)\alpha} - [e^c - (e^c - e^b)\alpha] \\ &= e^{.5-(.5-.2).495} - [e^{.5} - (e^{.5} - e^{.3}).495] \\ &= -0.0075\end{aligned}$$

$$\begin{aligned}\alpha_L^* &= \frac{-a}{b-a} + \frac{1}{b-a} \ln \left( \frac{e^b - e^a}{b-a} \right) \\ &= \frac{-.01}{.3-.01} + \frac{1}{.3-.01} \ln \left( \frac{e^{.3} - e^{.01}}{.3-.01} \right) \\ &= 0.490.\end{aligned}$$

Similarly,  $\varepsilon_L^* = \text{Exact value} - \text{Approx. value}$

$$\begin{aligned}&= e^{a+(b-a)\alpha} - [e^a + (e^b - e^a)\alpha] \\ &= e^{.01+(.3-.01).49} - [e^{.01} + (e^{.3} - e^{.01}).49] \\ &= -0.0123.\end{aligned}$$

So for this fairly large TPD failure intensity, as used in reliability calculations, the left deviation is about 12 in a left base of 290, or 4%, and the right deviation is about 8 in a right base of 200, also about 4%. Thus, even for these large failure intensity values the approximation error is small.

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Lewis Warren

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**Total number of copies: 30    Printed: 10    PDF: 20**

<b>DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION</b> <b>DOCUMENT CONTROL DATA</b>					
				1. PRIVACY MARKING/CAVEAT (OF DOCUMENT)	
2. TITLE  On Modelling Hybrid Uncertainty in Information				3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION)  Document (U) Title (U) Abstract (U)	
4. AUTHOR(S)  Lewis Warren				5. CORPORATE AUTHOR  DSTO Defence Science and Technology Organisation PO Box 1500 Edinburgh South Australia 5111 Australia	
6a. DSTO NUMBER DSTO-RR-0325		6b. AR NUMBER AR-013-848		6c. TYPE OF REPORT Research Report	7. DOCUMENT DATE February 2007
8. FILE NUMBER	9. TASK NUMBER JTW 05/147	10. TASK SPONSOR DGPlans		11. NO. OF PAGES 94	12. NO. OF REFERENCES 119
13. URL on the World Wide Web  <a href="http://www.dsto.defence.gov.au/corporate/reports/DSTO-RR-0325.pdf">http://www.dsto.defence.gov.au/corporate/reports/DSTO-RR-0325.pdf</a>				14. RELEASE AUTHORITY  Chief, Command and Control Division	
15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT  <p style="text-align: center;"><i>Approved for public release</i></p> <p>OVERSEAS ENQUIRIES OUTSIDE STATED LIMITATIONS SHOULD BE REFERRED THROUGH DOCUMENT EXCHANGE, PO BOX 1500, EDINBURGH, SA 5111</p>					
16. DELIBERATE ANNOUNCEMENT  No Limitations					
17. CITATION IN OTHER DOCUMENTS Yes					
18. DSTO RESEARCH LIBRARY THESAURUS  Uncertainty measures, Higher-order uncertainty, Hybrid uncertainty, Possibility Theory, Entropy					
19. ABSTRACT Numerical induction models are considered in this report to be models that aggregate lower-level information into higher-level measures for decision making. Various forms of uncertainty may be present in such models including hybrid uncertainties within the information elements being aggregated. After a review of some existing approaches for representing higher-order uncertainty in information, a new approach is presented to enable greater fidelity of uncertainty representation, and consequently more rigorous uncertainty management in aggregation operations. Several different applications then demonstrate the proposed procedures which have direct relevance to many Defence decision making models where higher-order uncertainty is ubiquitous. The overall objective of these procedures is to extract as much meaning from the input information as possible.					